On the Combination of Symbolic Constraints, Solution Domains, and Constraint Solvers*

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Abstract

When combining languages for symbolic constraints, one is typically faced with the problem of how to treat "mixed" constraints. The two main problems are (1) how to define a combined solution structure over which these constraints are to be solved, and (2) how to combine the constraint solving methods for pure constraints into one for mixed constraints. The paper introduces the notion of a "free amalgamated product" as a possible solution to the first problem. Subsequently, we define so-called *simply-combinable structures* (SC-structures). For SC-structures over disjoint signatures, a canonical amalgamation construction exists, which for the subclass of *strong* SC-structures yields the free amalgamated product. The combination technique of [BS92, BaS94a] can be used to combine constraint solvers for (strong) SC-structures over disjoint signatures into a solver for their (free) amalgamated product. In addition to term algebras modulo equational theories, the class of SC-structures contains many solution structures that have been used in constraint logic programming, such as the algebra of rational trees, feature structures, and domains consisting of hereditarily finite (wellfounded or non-wellfounded) nested sets and lists.

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Contents

1 Introduction

Many CLP dialects, and some of the related formalisms used in computational linguistics, provide for a combination of several "primitive" constraint languages. For example, in Prolog III [Col90], mixed constraints can be used to express lists of rational trees where some nodes can again be lists etc.; Mukai [Muk91] combines rational trees and record structures, and a domain that integrates rational trees and feature structures has been used in [SmT94]; Rounds [Rou88] introduces set-valued feature structures that interweave ordinary feature structures and non-wellfounded sets, and many other suggestions for integrating sets into logic programming exist [DOP91, DoR93].

In this paper, we study techniques for combining symbolic constraints from a more general point of view. On the practical side, these considerations may facilitate the design and implementation of new combined constraint languages and solvers. On the theoretical side, we hope to obtain a better understanding of the principles underlying existing combination methods. This should show their essential similarities and differences, and clarify their limitations.

When combining different constraint systems, at least three problems must be solved. The first problem, namely how to define the set of "mixed" constraints, is usually relatively trivial. The two remaining problems—which will be addressed in this paper—are

(1) how to define the *combined solution structure* over which the mixed constraints are to be solved, and

(2) once this combined structure is fixed, how to *combine constraint solvers* for the single languages in order to obtain a constraint solver for the mixed language.

The first part of this paper is concerned with the first aspect. So far, the problem of combining solution domains has not been discussed in a general and systematic way. The reason is that most of the general combination results obtained until now were concerned with cases where the solution structures are defined by logical theories. In this case, the combined structures are defined by the union of the theories. For example, in unification modulo equational theories, the single solution structures are term algebras $\mathcal{T}(\Sigma_1, X)/{=_{E_1}}$ and $\mathcal{T}(\Sigma_2, X)/{=_{E_2}}$ modulo equational theories E_1 and E_2. Thus, the obvious candidate for the combined structure is $\mathcal{T}(\Sigma_1 \cup \Sigma_2, X)/{=_{E_1 \cup E_2}}$, the term algebra modulo the union $E_1 \cup E_2$ of the theories. It is, however, easy to see that feature structures and the "non-wellfounded" solution domains (such as rational trees) mentioned above cannot be described as such quotient term algebras. For this reason, it is not a priori clear whether there is a canonical way of combining such structures. The same problem also arises for other solution domains of symbolic constraints.

As a possible solution to this problem, we introduce the abstract notion of a "free amalgamated product" of two arbitrary structures in Section 3. Whenever the free amalgamated product of two given structures $\mathcal{A}$ and $\mathcal{B}$ exists, it is unique up to isomorphism, and it is the most general element among all structures that can be considered as a reasonable combination of $\mathcal{A}$ and $\mathcal{B}$. For the case of quotient term algebras $\mathcal{T}(\Sigma_1, X)/{=_{E_1}}$ and $\mathcal{T}(\Sigma_2, X)/{=_{E_2}}$, the free amalgamated product yields the combined term algebra $\mathcal{T}(\Sigma_1 \cup \Sigma_2, X)/{=_{E_1 \cup E_2}}$. This shows that it makes sense to propose the free amalgamated product of two solution structures as an adequate combined solution structure.

With respect to the second problem–the problem of combining constraint solvers–rather general results have been obtained for unification in the union of equational theories over disjoint signatures [SS89, Bou90, BS92]. These results have been generalized to the case of signatures sharing constants [Rin92, KiR94], and to disunification [BaS93]. Prima facie, such an extension of results seems to be mainly an algorithmic problem. The difficulty, one might think, is to find the correct combination method. A closer look at the results reveals, however, that most of the recent combination algorithms use, modulo details, the same transformation steps.[1] In each case, the real problem is to show correctness of the "old" algorithm in the new situation. In [BaS94a] we have tried to isolate the essential algebraic and logical principles that guarantee that the—seemingly universal—combination scheme works. We found a simple and abstract algebraic condition—called combinability—that guarantees correctness of the combination scheme, and allows for a rather simple proof of this fact. In addition, it was shown that this condition characterizes the class of quotient term algebras (i.e., free algebras), or more generally (if additional predicates are present), the class of free structures. In the above mentioned proof, an explicit construction was given that can be used to amalgamate two quotient term algebras over disjoint signatures, and which yields the combined quotient term algebra as result.

In the second part of this paper it is shown that the concept of a combinable structure and the amalgamation construction can considerably be gene-

[1]Sometimes, additional steps are introduced just to adapt the general scheme to special situations (e.g., [KiR94, BaS93]). For optimization purposes, steps may be applied in different orders, and delay mechanisms are employed (e.g., [Bou90]).

ralized. This yields combination results that apply to most of the structures mentioned above, and which go far beyond the level of quotient term algebras. To this purpose, a weakened notion of "combinability" is introduced (Section 4). Structures that satisfy this weak form of combinability will be called *simply-combinable structures* (SC-structures).[2] The algebra of rational trees [Col84, Mah88], feature structures [APS94, SmT94], but also domains over hereditarily finite (wellfounded or non-wellfounded) nested sets and lists turn out to be SC-structures. The main difference between free structures (treated in [BaS94a]) and SC-structures is that free structures are generated by a (countably infinite) set of (free) generators, whereas this need not be the case for SC-structures (e.g., an infinite rational tree is not generated—in the algebraic sense—by its leaf nodes). This difference makes it necessary to give rather involved proofs for facts that are trivial for the case of free structures. Nevertheless, a variant of the amalgamation construction of [BaS94a] can be used to combine arbitrary SC-structures $\mathcal{A}$ and $\mathcal{B}$ over disjoint signatures Σ and Δ (Section 6). As a Σ-structure (resp. Δ-structure), the amalgam $\mathcal{A} \otimes \mathcal{B}$ is isomorphic to $\mathcal{A}$ (resp. $\mathcal{B}$). Consequently, pure Σ-constraints (resp. Δ-constraints) are solvable in $\mathcal{A}$ (resp. $\mathcal{B}$) iff they are solvable in $\mathcal{A} \otimes \mathcal{B}$. If $\mathcal{A}$ and $\mathcal{B}$ belong to the subclass of strong SC-structures, then it can be shown that $\mathcal{A} \otimes \mathcal{B}$ is in fact the free amalgamated product of $\mathcal{A}$ and $\mathcal{B}$ as defined in Section 3. In this case, the amalgamation construction can be applied iteratedly since $\mathcal{A} \otimes \mathcal{B}$ is again a strong SC-structure.

The combination scheme, in the form given in [BS92, BaS94a], can be used to combine constraint solvers for two arbitrary SC-structures $\mathcal{A}$ and $\mathcal{B}$ over disjoint signatures into a solver for $\mathcal{A} \otimes \mathcal{B}$ (Section 7). In this general setting, we consider *existential positive sentences* as constraints, and the constraint solvers are decision procedures for validity of such formulae in the given solution structure. Thus, decidability of the *existential* positive theory of $\mathcal{A} \otimes \mathcal{B}$ can be reduced to decidability of the positive theories of $\mathcal{A}$ and $\mathcal{B}$. For the case of strong SC-structures $\mathcal{A}$ and $\mathcal{B}$, the combination method can also treat *general positive sentences* (Section 8). Thus, in this case, decidability of the *full* positive theory of $\mathcal{A} \otimes \mathcal{B}$ can be reduced to decidability of the positive theories of $\mathcal{A}$ and $\mathcal{B}$. As one concrete application we show that validity of positive sentences is decidable in domains that interweave rational feature trees, (finite or rational) trees, hereditarily finite (wellfounded or non-wellfounded) sets, and hereditarily finite (wellfounded or non-wellfounded) lists.

[2] It has turned out that the notion of an SC-structure is closely related to the concept of a "unification algebra" [SS88], and to the notion of an "instantiation system" [Wil91].

2 Formal Preliminaries

A signature Σ consists of a finite set Σ_F of function symbols and a finite set Σ_P of predicate symbols, each of fixed arity. We assume that equality "=" is a logical constant that does not occur in Σ_P, and which is always interpreted as the identity relation. An atomic Σ-formula is an equation $s = t$ between Σ_F-terms s, t, or a relational formula $p[s_1, \ldots, s_m]$ where p is a predicate symbol in Σ_P of arity m and $s_1, \ldots, s_m$ are Σ_F-terms. A positive Σ-matrix is any Σ-formula obtained from atomic Σ-formulae using conjunction and disjunction only. A positive Σ-formula is obtained from a positive Σ-matrix by adding an arbitrary quantifier prefix, and an existential positive Σ-formula is a positive formula where the prefix consists of existential quantifiers only. Sentences are formulae without free variables. The notation $t(v_1, \ldots, v_n)$ (resp. $\varphi(v_1, \ldots, v_n)$) indicates that the set of all (free) variables of the term t (of the formula φ) forms a subset of $\{v_1, \ldots, v_n\}$. Letters $u, v, \ldots$ denote variables, and expressions $\vec{u}, \vec{v}, \ldots$ denote finite (possibly empty) sequences of variables.

A Σ-structure $\mathcal{A}^\Sigma$ has a non-empty carrier set A, and it interprets each $f \in \Sigma_F$ of arity n as an n-ary (total) function $f_\mathcal{A}$ on A, and each $p \in \Sigma_P$ of arity m as an m-ary relation $p_\mathcal{A}$ on A. Whenever we use a roman letter like A and an expression $\mathcal{A}^\Sigma$ in the same context, the former symbol denotes the carrier set of the Σ-structure denoted by the latter expression. For a formula $\varphi(v_1, \ldots, v_n)$ with free variables in $\{v_1, \ldots, v_n\}$, we write $\mathcal{A}^\Sigma \models \varphi(a_1, \ldots, a_n)$ to express that the formula φ is valid in $\mathcal{A}^\Sigma$ under the evaluation that maps v_i to $a_i \in A$ $(1 \leq i \leq n)$. Sometimes we will consider several signatures simultaneously. If Δ is a subset of the signature Σ, then any Σ-structure A^Σ can be considered as a Δ-structure (called the Δ-reduct of $\mathcal{A}^\Sigma$) by just forgetting about the interpretation of the additional symbols. In this situation, $\mathcal{A}^\Delta$ denotes the Δ-reduct of $\mathcal{A}^\Sigma$. Expressions $\vec{a}$ denote finite (possibly empty) sequences $\langle a_1, \ldots, a_k \rangle$ of elements of A. In order to simplify notation we will sometimes use $\vec{a}$ also to denote the set $\{a_1, \ldots, a_k\}$.

If $\mathcal{A}^\Sigma$ is a Σ-structure, each assignment $\nu : \mathit{Var} \to A$ has a unique extension to an evaluation $\hat{\nu}$ that maps each Σ-term $t = t(v_1, \ldots, v_n)$ to an element $\hat{\nu}(t) \in A$. An element $a \in A$ is *generated* by the subset A_0 of A if there exists a Σ-term $t = t(v_1, \ldots, v_n)$ and an assignment $\nu : \mathit{Var} \to A$ such that $\hat{\nu}(t) = a$ and $\nu(v_i) \in A_0$ for $i = 1, \ldots, n$. The subset A_1 of A is generated by $A_0 \subseteq A$ if every element $a \in A_1$ is generated by A_0.

A Σ-homomorphism is a mapping h between two structures $\mathcal{A}^\Sigma$ and $\mathcal{B}^\Sigma$

such that

$$\begin{aligned} h(f_{\mathcal{A}}(a_1,\ldots,a_n)) &= f_{\mathcal{B}}(h(a_1),\ldots,h(a_n)), \\ p_{\mathcal{A}}[a_1,\ldots,a_n] &\Rightarrow p_{\mathcal{B}}[h(a_1),\ldots,h(a_n)] \end{aligned}$$

for all $f \in \Sigma_F$, $p \in \Sigma_P$, and $a_1, \ldots, a_n \in A$. Letters $h, g, \ldots$, possibly with subscript, denote homomorphisms. Whenever the signature Σ is not clear from the context, expressions $h^\Sigma, g^\Sigma, \ldots$ will be used. A Σ-isomorphism is a bijective Σ-homomorphism $h : \mathcal{A}^\Sigma \to \mathcal{B}^\Sigma$ such that

$$p_{\mathcal{A}}[a_1,\ldots,a_n] \Longleftrightarrow p_{\mathcal{B}}[h(a_1),\ldots,h(a_n)],$$

for all $p \in \Sigma_P$, and all $a_1, \ldots, a_n \in A$. Equivalently, one can require that the inverse mapping h^{-1} is also homomorphic.

As a matter of fact, validity of arbitrary formulae is preserved under isomorphisms. There is a less trivial connection between surjective homomorphisms and positive formulae, which will become important in the proof of correctness of our method for combining constraint solvers (see [Mal73], pp. 143, 144 for a proof).

Lemma 2.1 *Let $h : \mathcal{A}^\Sigma \to \mathcal{B}^\Sigma$ be a surjective homomorphism between the Σ-structures $\mathcal{A}^\Sigma$ and $\mathcal{B}^\Sigma$, $\varphi(v_1, \ldots, v_m)$ be a positive Σ-formula, and $a_1, \ldots, a_m$ be elements of A. Then $\mathcal{A}^\Sigma \models \varphi(a_1, \ldots, a_m)$ implies $\mathcal{B}^\Sigma \models \varphi(h(a_1), \ldots, h(a_m))$.*

Since validity of existential formulae is preserved in a superstructure (see, e.g., [Mal71] pp.) the following variant of Lemma 2.1 for arbitrary homomorphisms follows.

Lemma 2.2 *Let $h : \mathcal{A}^\Sigma \to \mathcal{B}^\Sigma$ be a homomorphism between the Σ-structures $\mathcal{A}^\Sigma$ and $\mathcal{B}^\Sigma$, $\varphi(v_1, \ldots, v_m)$ be an existential positive Σ-formula, and $a_1, \ldots, a_m$ be elements of A. Then $\mathcal{A}^\Sigma \models \varphi(a_1, \ldots, a_m)$ implies $\mathcal{B}^\Sigma \models \varphi(h(a_1), \ldots, h(a_m))$.*

A Σ-endomorphism of $\mathcal{A}^\Sigma$ is a homomorphism $h^\Sigma : \mathcal{A}^\Sigma \to \mathcal{A}^\Sigma$. With $End_{\mathcal{A}}^\Sigma$ we denote the monoid of all endomorphisms of the Σ-structure $\mathcal{A}^\Sigma$, with composition as operation. The notation $\mathcal{M} \leq End_{\mathcal{A}}^\Sigma$ expresses that $\mathcal{M}$ is a submonoid of $End_{\mathcal{A}}^\Sigma$.

If $g : A \to B$ and $h : B \to C$ are mappings, then $g \circ h : A \to C$ denotes their composition. Note that $g \circ h$ means that g is applied first, and then h. Let $g_1 : A \to C$ and $g_2 : B \to D$ be two mappings. We say that g_1 *and* g_2

coincide on $E \subseteq A \cap B$ if $g_1(e) = g_2(e)$ for all $e \in E$. For a set A, we denote the identity mapping on A by Id_A. If A is the carrier of a Σ-structure $\mathcal{A}$, then Id_A is the unit of the monoid $End_{\mathcal{A}}^{\Sigma}$.

Given a signature Σ, "constraints" are usually introduced as Σ-formulae (of a particular syntactic type) $\varphi(v_1, \ldots, v_n)$ with free variables. The constraint $\varphi(v_1, \ldots, v_n)$ is solvable in the structure $\mathcal{A}^{\Sigma}$ iff there are $a_1, \ldots, a_n \in A$ such that $\mathcal{A}^{\Sigma} \models \varphi(a_1, \ldots, a_n)$. Thus solvability of φ in $\mathcal{A}^{\Sigma}$ and validity of the sentence $\exists v_1 \ldots \exists v_n\ \varphi(v_1, \ldots, v_n)$ in $\mathcal{A}^{\Sigma}$ are equivalent. In this paper we shall always use the second point of view. As constraints we consider existential positive and positive sentences. We are mainly interested in solving "mixed" constraints. This means that we consider two different signatures Σ and Δ, with fixed solution structures $\mathcal{B}_1^{\Sigma}$ and $\mathcal{B}_2^{\Delta}$. A mixed constraint is a positive (or existential positive) $(\Sigma \cup \Delta)$-sentence. Thus, one needs a $(\Sigma \cup \Delta)$-structure as solution structure. Obviously, if we want to reduce solvability of mixed constraints to solvability of pure Σ_i-constraints in the Σ_i-structures $\mathcal{B}_i$ $(i = 1, 2)$, this "combined" solution structure should be in an appropriate relationship with the single structures $\mathcal{B}_1^{\Sigma}$ and $\mathcal{B}_2^{\Delta}$.

3 Combination of Structures

Suppose that $\mathcal{B}_1^{\Sigma}$ and $\mathcal{B}_2^{\Delta}$ are two structures. In the first part of this section we shall discuss the following question: What conditions should a $(\Sigma \cup \Delta)$-structure $\mathcal{C}^{\Sigma \cup \Delta}$ satisfy to be called a "combination" of $\mathcal{B}_1^{\Sigma}$ and $\mathcal{B}_2^{\Delta}$? This will lead to the definition of the free amalgamated product. In the second part of the section, we shall show that, under certain restrictions, the product construction is associative.

3.1 The free amalgamated product

The central definition of this section will be obtained after three steps, each introducing a restriction that is motivated by the example of the combination of term algebras modulo equational theories. The structures $\mathcal{B}_1^{\Sigma}$ and $\mathcal{B}_2^{\Delta}$ will be called the *components* in the sequel.

"**Restriction 1:**" *Homomorphisms that "embed" the components into the combined structure must exist. If the components share a common substructure, then the homomorphisms must agree on this substructure.*

In fact, a minimal requirement seems to be that both structures must in some sense be embedded in their combination. It would, however, be too restrictive to demand that the components are substructures of the combined structure. For the case of consistent equational theories E, F over disjoint signatures Σ, Δ, there exist 1–1-embeddings of $\mathcal{T}(\Sigma, V)/{=_E}$ and $\mathcal{T}(\Delta, V)/{=_F}$ into $\mathcal{T}(\Sigma \cup \Delta, V)/{=_{E \cup F}}$. For non-disjoint signatures, however, these "embeddings" need no longer be 1–1. Note that even for disjoint signatures Σ and Δ there is a common part, namely the trivial structure represented by the set V of variables. A reasonable requirement is that elements of the common part are mapped to the same element of the combined structure by the homomorphic embeddings. To be as general as possible, we do not assume that the "common part" is really a substructure of $\mathcal{B}_1^\Sigma$ and $\mathcal{B}_2^\Delta$. Instead, we assume that it is just homomorphically embedded in both structures. Restriction 1 motivates the following definition.

Definition 3.1 *Let Σ and Δ be signatures, and let $\Gamma \subseteq \Sigma \cap \Delta$. A triple $(\mathcal{A}^\Gamma, \mathcal{B}_1^\Sigma, \mathcal{B}_2^\Delta)$ with given homomorphic embeddings*

$$h^\Gamma_{A-B_1} : \mathcal{A}^\Gamma \to \mathcal{B}_1^\Sigma \quad \textit{and} \quad h^\Gamma_{A-B_2} : \mathcal{A}^\Gamma \to \mathcal{B}_2^\Delta$$

is called an amalgamation base. *The structure $\mathcal{D}^{\Sigma\cup\Delta}$* closes *the amalgamation base $(\mathcal{A}^\Gamma, \mathcal{B}_1^\Sigma, \mathcal{B}_2^\Delta)$ iff there are homomorphisms*

$$h^\Sigma_{B_1-D} : \mathcal{B}_1^\Sigma \to \mathcal{D}^\Sigma \quad \textit{and} \quad h^\Delta_{B_2-D} : \mathcal{B}_2^\Delta \to \mathcal{D}^\Delta$$

such that $h^\Gamma_{A-B_1} \circ h^\Sigma_{B_1-D} = h^\Gamma_{A-B_2} \circ h^\Delta_{B_2-D}$. We call $(\mathcal{D}^{\Sigma\cup\Delta}, h^\Sigma_{B_1-D}, h^\Delta_{B_2-D})$ an amalgamated product *of $(\mathcal{A}^\Gamma, \mathcal{B}_1^\Sigma, \mathcal{B}_2^\Delta)$.*

If the "embedding" homomorphisms are irrelevant or clear from the context, we shall also call the structure $\mathcal{D}^{\Sigma\cup\Delta}$ an amalgamated product of $\mathcal{B}_1^\Sigma$ and $\mathcal{B}_2^\Delta$ over $\mathcal{A}^\Gamma$. It should be clear that it is not reasonable to accept an arbitrary amalgamated product as the combined structure of $\mathcal{B}_1^\Sigma$ and $\mathcal{B}_2^\Delta$.

"**Restriction 2:"** *The combined structure should share "relevant" structural properties with the components.*

This principle accounts for the fact that there must be some kind of (logical, algebraic, algorithmic) relationship between the components and the combined structure. In the case of quotient term algebras $\mathcal{T}(\Sigma, V)/{=_E}$ and $\mathcal{T}(\Delta, V)/{=_F}$, the combined algebra $\mathcal{T}(\Sigma \cup \Delta, V)/{=_{E\cup F}}$ satisfies $E \cup F$. In general, we cannot use this as a condition on the structures that close the amalgamation base since we need not have theories defining $\mathcal{B}_1^\Sigma$ and $\mathcal{B}_2^\Delta$. However, for the case of quotient term algebras there is an equivalent algebraic reformulation:

Proposition 3.2 *For a* $(\Sigma \cup \Delta)$*-algebra* $\mathcal{C}^{\Sigma\cup\Delta}$ *and a countably infinite set (of variables)* V*, the following conditions are equivalent:*

1. *The structure* $\mathcal{C}^{\Sigma\cup\Delta}$ *satisfies all axioms of* $E \cup F$*.*

2. *For every mapping* $g_{V-C} : V \to C$ *there exist unique homomorphisms* $h^{\Sigma}_{T_1-C} : \mathcal{T}(\Sigma, V)/_{=_E} \to \mathcal{C}^{\Sigma}$ *and* $h^{\Delta}_{T_2-C} : \mathcal{T}(\Delta, V)/_{=_F} \to \mathcal{C}^{\Delta}$ *extending* g_{V-C}*.*

Proof. First, we show "1 $\to$ 2." Since the $(\Sigma \cup \Delta)$-algebra $\mathcal{C}^{\Sigma\cup\Delta}$ satisfies $E \cup F$, its Σ-reduct $\mathcal{C}^{\Sigma}$ satisfies E and its Δ-reduct $\mathcal{C}^{\Delta}$ satisfies F. Thus, existence and uniqueness of the desired homomorphisms follows from the fact that $\mathcal{T}(\Sigma, V)/_{=_E}$ is free over V for the class of all models of E, and $\mathcal{T}(\Delta, V)/_{=_F}$ is free over V for the class of all models of F.

In order to show "2 $\to$ 1," assume that $\mathcal{C}^{\Sigma\cup\Delta}$ satisfies the algebraic characterization (2). Let $s(v_1, \ldots, v_n) = t(v_1, \ldots, v_n)$ be an equation in $E \cup F$, where the variables $v_1, \ldots, v_n$ occurring in $s = t$ are (without loss of generality) assumed to be in V. Now, assume that $\mathcal{C}^{\Sigma\cup\Delta}$ does not satisfy $s = t$. Thus, there exist elements $c_1, \ldots, c_n$ of C such that

$$\mathcal{C}^{\Sigma\cup\Delta} \not\models s(c_1, \ldots, c_n) = t(c_1, \ldots, c_n).$$

Without loss of generality, we assume that $s = t$ is an equation in E. Let $g : V \to C$ be a mapping such that $g(v_i) = c_i$ (for $1 \leq i \leq n$). By (2), there exists a homomorphism $h^{\Sigma} : \mathcal{T}(\Sigma, V)/_{=_E} \to \mathcal{C}^{\Sigma}$ that extends g. However, $s = t \in E$ implies $s =_E t$, and thus s and t belong to the same $=_E$-class in $\mathcal{T}(\Sigma, V)/_{=_E}$. This shows that $h(s) = h(t)$, which contradicts our assumption that $\mathcal{C}^{\Sigma\cup\Delta} \not\models s(c_1, \ldots, c_n) = t(c_1, \ldots, c_n)$. $\square$

In Section 6 we shall restrict the admissible structures for closing an amalgamation base $(\mathcal{A}^{\Gamma}, \mathcal{B}_1^{\Sigma}, \mathcal{B}_2^{\Delta})$ to structures satisfying the second condition of the proposition. In the remainder of this section it is sufficient to assume that some class $Adm(\mathcal{B}_1^{\Sigma}, \mathcal{B}_2^{\Delta})$ of *admissible* structures for closing the amalgamation base has been fixed.

Definition 3.3 *Let* $(\mathcal{A}^{\Gamma}, \mathcal{B}_1^{\Sigma}, \mathcal{B}_2^{\Delta})$ *be an amalgamation base, let* $\mathrm{Adm}(\mathcal{B}_1^{\Sigma}, \mathcal{B}_2^{\Delta})$ *be a class of* $(\Sigma \cup \Delta)$*-structures, to be called* admissible structures*. An amalgamated product* $(\mathcal{D}^{\Sigma\cup\Delta}, h^{\Sigma}_{B_1-D}, h^{\Delta}_{B_2-D})$ *of* $(\mathcal{A}^{\Gamma}, \mathcal{B}_1^{\Sigma}, \mathcal{B}_2^{\Delta})$ *is called* admissible with respect to $Adm(\mathcal{B}_1^{\Sigma}, \mathcal{B}_2^{\Delta})$ *(or simply* admissible*, if the class of admissible structures is clear from the context) iff* $\mathcal{D}^{\Sigma\cup\Delta} \in Adm(\mathcal{B}_1^{\Sigma}, \mathcal{B}_2^{\Delta})$*.*

In the case of term algebras, the combined algebra $\mathcal{T}(\Sigma \cup \Delta, V)/_{=_{E\cup F}}$ is not just any algebra satisfying $E \cup F$—it is the free algebra.

"**Restriction 3:**" *Whenever possible, we want to obtain a most general element among all admissible amalgamated products of the components.*

This motivates the definition of the free amalgamated product by a universal property that is similar to the one of free algebras.

Definition 3.4 *Let* $(\mathcal{A}^\Gamma, \mathcal{B}_1^\Sigma, \mathcal{B}_2^\Delta)$ *be an amalgamation base, and assume that* $Adm(\mathcal{B}_1^\Sigma, \mathcal{B}_2^\Delta)$ *is the class of admissible* $(\Sigma \cup \Delta)$*-structures. The admissible amalgamated product* $(\mathcal{C}^{\Sigma\cup\Delta}, h^\Sigma_{B_1-C}, h^\Delta_{B_2-C})$ *of* $\mathcal{B}_1^\Sigma$ *and* $\mathcal{B}_2^\Delta$ *over* $\mathcal{A}^\Gamma$ *is called a* free amalgamated product with respect to $Adm(\mathcal{B}_1^\Sigma, \mathcal{B}_2^\Delta)$ *iff for every admissible amalgamated product* $(\mathcal{D}^{\Sigma\cup\Delta}, h^\Sigma_{B_1-D}, h^\Delta_{B_2-D})$ *of* $\mathcal{B}_1^\Sigma$ *and* $\mathcal{B}_2^\Delta$ *over* $\mathcal{A}^\Gamma$ *there exists a* unique *homomorphism* $h^{\Sigma\cup\Delta}_{C-D} : \mathcal{C}^{\Sigma\cup\Delta} \to \mathcal{D}^{\Sigma\cup\Delta}$ *such that*

$$h^\Sigma_{B_1-D} = h^\Sigma_{B_1-C} \circ h^{\Sigma\cup\Delta}_{C-D} \quad \text{and} \quad h^\Delta_{B_2-D} = h^\Delta_{B_2-C} \circ h^{\Sigma\cup\Delta}_{C-D}.$$

$$\begin{tikzcd}
 & \mathcal{B}_1^\Sigma \arrow[dr, "h_{B_1-C}"] \arrow[drr, "h_{B_1-D}"] & & \\
\mathcal{A}^\Gamma \arrow[ur, "h_{A-B_1}"] \arrow[dr, "h_{A-B_2}"'] & & \mathcal{C}^{\Sigma\cup\Delta} \arrow[r, "h_{C-D}"] & \mathcal{D}^{\Sigma\cup\Delta} \\
 & \mathcal{B}_2^\Delta \arrow[ur, "h_{B_2-C}"] \arrow[urr, "h_{B_2-D}"'] & &
\end{tikzcd}$$

Free amalgamated products need not exist, but if they exist they are unique up to isomorphism.

Theorem 3.5 *Let* $(\mathcal{A}^\Gamma, \mathcal{B}_1^\Sigma, \mathcal{B}_2^\Delta)$ *be an amalgamation base with fixed homomorphic embeddings* $h^\Gamma_{A-B_1} : \mathcal{A}^\Gamma \to \mathcal{B}_1^\Sigma$ *and* $h^\Gamma_{A-B_2} : \mathcal{A}^\Gamma \to \mathcal{B}_2^\Delta$*. The free amalgamated product of* $\mathcal{B}_1^\Sigma$ *and* $\mathcal{B}_2^\Delta$ *over* $\mathcal{A}^\Gamma$ *with respect to a given class* $Adm(\mathcal{B}_1^\Sigma, \mathcal{B}_2^\Delta)$ *is unique up to* $(\Sigma \cup \Delta)$*-isomorphism.*

Proof. Let $\mathcal{C}^{\Sigma\cup\Delta}$ and $\mathcal{D}^{\Sigma\cup\Delta}$ be free amalgamated products of $\mathcal{B}_1^\Sigma$ and $\mathcal{B}_2^\Delta$ over $\mathcal{A}^\Gamma$ with respect to $Adm(\mathcal{B}_1^\Sigma, \mathcal{B}_2^\Delta)$. It follows that both structures belong to the class of admissible structures $Adm(\mathcal{B}_1^\Sigma, \mathcal{B}_2^\Delta)$. Since $\mathcal{C}^{\Sigma\cup\Delta}$ is an amalgamated product, there exist homomorphisms $h^\Sigma_{B_1-C} : \mathcal{B}_1^\Sigma \to \mathcal{C}^\Sigma$ and $h^\Delta_{B_2-C} : B_2^\Delta \to \mathcal{C}^\Delta$ such that $h^\Gamma_{A-B_1} \circ h^\Sigma_{B_1-C} = h^\Gamma_{A-B_2} \circ h^\Delta_{B_2-C}$. Similarly there exist homomorphisms $h^\Sigma_{B_1-D} : \mathcal{B}_1^\Sigma \to \mathcal{D}^\Sigma$ and $h^\Delta_{B_2-D} : B_2^\Delta \to \mathcal{D}^\Delta$ such that $h^\Gamma_{A-B_1} \circ h^\Sigma_{B_1-D} = h^\Gamma_{A-B_2} \circ h^\Delta_{B_2-D}$.

Since $\mathcal{C}^{\Sigma\cup\Delta}$ is a free amalgamated product, there exists a unique homomorphism $f^{\Sigma\cup\Delta}_{C-D} : \mathcal{C}^{\Sigma\cup\Delta} \to \mathcal{D}^{\Sigma\cup\Delta}$ such that

$$h^\Sigma_{B_1-D} = h^\Sigma_{B_1-C} \circ f^{\Sigma\cup\Delta}_{C-D} \quad \text{and} \quad h^\Delta_{B_2-D} = h^\Delta_{B_2-C} \circ f^{\Sigma\cup\Delta}_{C-D}.$$

Similarly, there exists a unique homomorphism $f^{\Sigma\cup\Delta}_{D-C} : \mathcal{D}^{\Sigma\cup\Delta} \to \mathcal{C}^{\Sigma\cup\Delta}$ such that

$$h^{\Sigma}_{B_1-C} = h^{\Sigma}_{B_1-D} \circ f^{\Sigma\cup\Delta}_{D-C} \quad \text{and} \quad h^{\Delta}_{B_2-C} = h^{\Delta}_{B_2-D} \circ f^{\Sigma\cup\Delta}_{D-C}.$$

This implies $h^{\Sigma}_{B_1-C} = h^{\Sigma}_{B_1-D} \circ f^{\Sigma\cup\Delta}_{D-C} = h^{\Sigma}_{B_1-C} \circ f^{\Sigma\cup\Delta}_{C-D} \circ f^{\Sigma\cup\Delta}_{D-C}$, and similarly we obtain $h^{\Delta}_{B_2-C} = h^{\Delta}_{B_2-C} \circ f^{\Sigma\cup\Delta}_{C-D} \circ f^{\Sigma\cup\Delta}_{D-C}$.

Since $\mathcal{C}^{\Sigma\cup\Delta}$ is a *free* amalgamated product, and since $\mathcal{C}^{\Sigma\cup\Delta} \in Adm(\mathcal{B}^{\Sigma}_1, \mathcal{B}^{\Delta}_2)$, there exists a unique $(\Sigma \cup \Delta)$-endomorphism $h^{\Sigma\cup\Delta}$ of $\mathcal{C}^{\Sigma\cup\Delta}$ such that

$$\begin{aligned} h^{\Sigma}_{B_1-C} &= h^{\Sigma}_{B_1-C} \circ h^{\Sigma\cup\Delta} \\ h^{\Delta}_{B_2-C} &= h^{\Delta}_{B_2-C} \circ h^{\Sigma\cup\Delta}. \end{aligned}$$

We have just seen that $f^{\Sigma\cup\Delta}_{C-D} \circ f^{\Sigma\cup\Delta}_{D-C}$ satisfies these properties, and obviously, Id_C satisfies them as well. This shows that $f^{\Sigma\cup\Delta}_{C-D} \circ f^{\Sigma\cup\Delta}_{D-C} = Id_C$. Symmetrically, one can also show $f^{\Sigma\cup\Delta}_{D-C} \circ f^{\Sigma\cup\Delta}_{C-D} = Id_D$.

To sum up, we have shown that $f^{\Sigma\cup\Delta}_{C-D}$ and $f^{\Sigma\cup\Delta}_{D-C}$ are isomorphisms that are inverse to each other. □

The theorem justifies to speak about *the* free amalgamated product of two structures (provided that the embedding homomorphisms and the class of admissible structures are fixed). In this situation, we shall often write $\mathcal{B}_1 \odot \mathcal{B}_2$ for the free amalgamated product of $\mathcal{B}_1$ and $\mathcal{B}_2$.

In Section 6 we shall give an explicit construction of the free amalgamated product for the class of "strong SC-structures." For our standard example, term algebras modulo equational theories, the free amalgamated product yields the combined quotient term algebra, which shows that the definition of the free amalgamated product makes sense.

Proposition 3.6 *Let $\mathcal{B}^{\Sigma}_1 = \mathcal{T}(\Sigma, V)/_{=_E}$ and $\mathcal{B}^{\Delta}_2 = \mathcal{T}(\Delta, V)/_{=_F}$ for consistent equational theories E and F. Let $\mathrm{Adm}(\mathcal{B}^{\Sigma}_1, \mathcal{B}^{\Delta}_2)$ be the class of algebras satisfying (one of) the conditions of Proposition 3.2. For the amalgamation base $(\mathcal{T}(\Sigma \cap \Delta, V), \mathcal{B}^{\Sigma}_1, \mathcal{B}^{\Delta}_2)$, the free amalgamated product with respect to $Adm(\mathcal{B}^{\Sigma}_1, \mathcal{B}^{\Delta}_2)$ is isomorphic to the combined algebra $\mathcal{T}(\Sigma \cup \Delta, V)/_{=_{E\cup F}}$.*

Proof. Since $\mathcal{C}^{\Sigma\cup\Delta} := \mathcal{T}(\Sigma \cup \Delta, V)/_{=_{E\cup F}}$ satisfies all axioms of $E \cup F$, it is clearly an admissible algebra in $Adm(\mathcal{B}^{\Sigma}_1, \mathcal{B}^{\Delta}_2)$. The Σ-reduct $\mathcal{C}^{\Sigma}$ of $\mathcal{C}^{\Sigma\cup\Delta}$ satisfies E, and the Δ-reduct $\mathcal{C}^{\Delta}$ satisfies F. Since $\mathcal{B}^{\Sigma}_1$ is free over V for the class of all models of E, there exists a unique Σ-homomorphism $h^{\Sigma}_{B_1-C} : \mathcal{B}^{\Sigma}_1 \to \mathcal{C}^{\Sigma}$ that extends Id_V. Similarly, there exists a unique Δ-homomorphism $h^{\Delta}_{B_2-C} : \mathcal{B}^{\Delta}_2 \to \mathcal{C}^{\Delta}$ extending Id_V.

In addition, since $\mathcal{A}^\Gamma := \mathcal{T}(\Sigma \cap \Delta, V)$ is the (absolutely) free Γ-algebra, there exist unique homomorphisms $h^\Gamma_{A-B_1} : \mathcal{A}^\Gamma \to \mathcal{B}_1^\Gamma$ and $h^\Gamma_{A-B_2} : \mathcal{A}^\Gamma \to \mathcal{B}_2^\Gamma$ extending Id_V. It follows that

$$h^\Gamma_{A-B_1} \circ h^\Sigma_{B_1-C} = h^\Gamma_{A-B_2} \circ h^\Delta_{B_2-C},$$

since both homomorphisms represent the unique extension of Id_V to a Γ-homomorphism $\mathcal{A}^\Gamma \to \mathcal{C}^\Gamma$. Thus, we have shown that $\mathcal{C}^{\Sigma\cup\Delta}$ is in fact an admissible amalgamated product of $\mathcal{B}_1^\Sigma$ and $\mathcal{B}_2^\Delta$ over $\mathcal{A}^\Gamma$ with respect to $Adm(\mathcal{B}_1^\Sigma, \mathcal{B}_2^\Delta)$.

In order to show that it is free, assume that $\mathcal{D}^{\Sigma\cup\Delta}$ is an admissible algebra in $Adm(\mathcal{B}_1^\Sigma, \mathcal{B}_2^\Delta)$, and that homomorphisms $h^\Sigma_{B_1-D} : \mathcal{B}_1^\Sigma \to \mathcal{D}^{\Sigma\cup\Delta}$ and $h^\Delta_{B_2-D} : \mathcal{B}_2^\Delta \to \mathcal{D}^{\Sigma\cup\Delta}$ satisfying

$$h^\Gamma_{A-B_1} \circ h^\Sigma_{B_1-D} = h^\Gamma_{A-B_2} \circ h^\Delta_{B_2-D}$$

are given. Let $f_0 : V \to D$ be the restriction of $h^\Gamma_{A-B_1} \circ h^\Sigma_{B_1-D} = h^\Gamma_{A-B_2} \circ h^\Delta_{B_2-D}$ to V. Since $\mathcal{D}^{\Sigma\cup\Delta}$ is an admissible structure, it satisfies all axioms of $E \cup F$, and since $\mathcal{C}^{\Sigma\cup\Delta}$ is free over V for the class of all models of $E \cup F$, the mapping $f_0 : V \to D$ has a unique extension to a homomorphism $f^{\Sigma\cup\Delta}_{C-D} : \mathcal{C}^{\Sigma\cup\Delta} \to \mathcal{D}^{\Sigma\cup\Delta}$.

Since h_{B_1-C} and h_{A-B_1} coincides with Id_V on V, $h^\Sigma_{B_1-C} \circ f^{\Sigma\cup\Delta}_{C-D}$ and $h^\Sigma_{B_1-D}$ are two Σ-homomorphisms $\mathcal{B}_1^\Sigma \to \mathcal{D}^\Sigma$ that coincide on V. Thus $h^\Sigma_{B_1-C} \circ f^{\Sigma\cup\Delta}_{C-D} = h^\Sigma_{B_1-D}$, since $\mathcal{B}_1^\Sigma$ is free over V for the class of all models of E, and the Σ-reduct $\mathcal{D}^\Sigma$ of $\mathcal{D}^{\Sigma\cup\Delta}$ satisfies E. Similarly, one can prove that $h^\Delta_{B_2-C} \circ f^{\Sigma\cup\Delta}_{C-D} = h^\Delta_{B_2-D}$.

It remains to be shown that $f^{\Sigma\cup\Delta}_{C-D}$ is unique with this property. Since h_{B_1-C} coincides with Id_V on V, any $(\Sigma \cup \Delta)$-homomorphism $f : \mathcal{C}^{\Sigma\cup\Delta} \to \mathcal{D}^{\Sigma\cup\Delta}$ satisfying $h^\Sigma_{B_1-C} \circ f = h^\Sigma_{B_1-D}$ coincides with $h^\Sigma_{B_1-D}$ on V. Since $C^{\Sigma\cup\Delta}$ is free, there can be only one such homomorphism. $\square$

Notions of "amalgamated product," similar to the one given above, can be found in universal algebra, model theory, and in category theory (see, e.g., [Mal73, Che76, DG93]). There are, however, certain differences between our situation and the typical situations in which amalgamation occurs in other areas. In algebra or model theory, amalgamation has been introduced for *particular classes of algebraic structures* such as groups, fields, skew fields etc. Amalgamation is studied for such a fixed class of structures over the same signature, and it is assumed that these structures all satisfy the same set of axioms (e.g., those for groups, fields, skew fields, etc.). In our case, algebras over different signatures are amalgamated, and these algebras satisfy different types of axioms (or are not defined by axioms at all).

3.2 Associativity of free amalgamation

The product construction is obviously commutative if the definition of the class of admissible structures satisfies $Adm(\mathcal{B}_1^\Sigma, \mathcal{B}_2^\Delta) = Adm(\mathcal{B}_2^\Delta, \mathcal{B}_1^\Sigma)$. In order to obtain associativity as well, we need some additional conditions on the class of admissible structures.

Before formulating these restrictions, we extend the definition of an amalgamation base and of the free amalgamated product to the case of three structures. Let $\Gamma \subseteq \Sigma_1 \cap \Sigma_2 \cap \Sigma_3$. A quadruple $(\mathcal{A}^\Gamma, \mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2}, \mathcal{B}_3^{\Sigma_3})$ with given homomorphic embeddings

$$h_{A-B_i}^\Gamma : \mathcal{A}^\Gamma \to \mathcal{B}_i^{\Sigma_i} \quad (i = 1, 2, 3)$$

is called a *simultaneous amalgamation base*. The structure $\mathcal{D}^{\Sigma_1 \cup \Sigma_2 \cup \Sigma_3}$ *closes* the simultaneous amalgamation base $(\mathcal{A}^\Gamma, \mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2}, \mathcal{B}_3^{\Sigma_3})$ iff, for $i = 1, 2, 3$, there are homomorphisms

$$h_{B_i-D}^{\Sigma_i} : \mathcal{B}_i^{\Sigma_i} \to \mathcal{D}^{\Sigma_i}$$

such that $h_{A-B_1}^\Gamma \circ h_{B_1-D}^{\Sigma_1} = h_{A-B_2}^\Gamma \circ h_{B_2-D}^{\Sigma_2} = h_{A-B_3}^\Gamma \circ h_{B_3-D}^{\Sigma_3}$. In this case, $(\mathcal{D}^{\Sigma_1 \cup \Sigma_2 \cup \Sigma_3}, h_{B_1-D}^{\Sigma_1}, h_{B_2-D}^{\Sigma_2}, h_{B_3-D}^{\Sigma_3})$ is a *simultaneous amalgamated product* of $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ over $\mathcal{A}^\Gamma$.

Now, assume that a class of admissible structures $Adm(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$ is fixed. The simultaneous amalgamated product $(\mathcal{C}^{\Sigma_1 \cup \Sigma_2 \cup \Sigma_3}, h_{B_1-C}^{\Sigma_1}, h_{B_2-C}^{\Sigma_2}, h_{B_3-C}^{\Sigma_3})$ is called *admissible* iff $\mathcal{C}^{\Sigma_1 \cup \Sigma_2 \cup \Sigma_3} \in Adm(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$. The admissible simultaneous amalgamated product $(\mathcal{C}^{\Sigma_1 \cup \Sigma_2 \cup \Sigma_3}, h_{B_1-C}^{\Sigma_1}, h_{B_2-C}^{\Sigma_2}, h_{B_3-C}^{\Sigma_3})$ of $\mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2}, \mathcal{B}_3^{\Sigma_3}$ over $\mathcal{A}^\Gamma$ is called a *free simultaneous amalgamated product with respect to* $Adm(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$ iff for every admissible simultaneous amalgamated product $(\mathcal{D}^{\Sigma_1 \cup \Sigma_2 \cup \Sigma_3}, h_{B_1-D}^{\Sigma_1}, h_{B_2-D}^{\Sigma_2}, h_{B_3-D}^{\Sigma_3})$ there exists a *unique* homomorphism

$$f_{C-D}^{\Sigma_1 \cup \Sigma_2 \cup \Sigma_3} : \mathcal{C}^{\Sigma_1 \cup \Sigma_2 \cup \Sigma_3} \to \mathcal{D}^{\Sigma_1 \cup \Sigma_2 \cup \Sigma_3}$$

such that for all $i = 1, 2, 3$,

$$g_{B_i-D}^{\Sigma_i} = h_{B_i-C}^{\Sigma_i} \circ f_{C-D}^{\Sigma_1 \cup \Sigma_2 \cup \Sigma_3}.$$

As for the binary free amalgamated product, one can show that the free simultaneous amalgamated product is unique up to isomorphism, provided that it exists. For this reason, associativity of the free amalgamated product (under certain restrictions) is an easy consequence of the next lemma and its dual.

Lemma 3.7 *Let $\Gamma \subseteq \Sigma_1 \cap \Sigma_2 \cap \Sigma_3$, and let $\mathcal{A}^\Gamma, \mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2}, \mathcal{B}_3^{\Sigma_3}$ be structures with fixed homomorphic embeddings $h^\Gamma_{A-B_1} : \mathcal{A}^\Gamma \to \mathcal{B}_1^{\Sigma_1}$, $h^\Gamma_{A-B_2} : \mathcal{A}^\Gamma \to \mathcal{B}_2^{\Sigma_2}$, and $h^\Gamma_{A-B_3} : \mathcal{A}^\Gamma \to \mathcal{B}_3^{\Sigma_3}$. Assume that the free amalgamated product $\mathcal{B}_2 \odot \mathcal{B}_3$ of $\mathcal{B}_2$ and $\mathcal{B}_3$, and the free amalgamated product $\mathcal{B}_1 \odot (\mathcal{B}_2 \odot \mathcal{B}_3)$ of $\mathcal{B}_1$ and $\mathcal{B}_2 \odot \mathcal{B}_3$ exist, and that the classes of admissible structures satisfy*

$$\mathcal{B}_1 \odot (\mathcal{B}_2 \odot \mathcal{B}_3) \in Adm(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3) \subseteq Adm(\mathcal{B}_2, \mathcal{B}_3) \cap Adm(\mathcal{B}_1, \mathcal{B}_2 \odot \mathcal{B}_3).$$

Then $\mathcal{B}_1 \odot (\mathcal{B}_2 \odot \mathcal{B}_3)$ is the free simultaneous amalgamated product of $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$ over $\mathcal{A}^\Gamma$.

Proof. Let $\mathcal{B}_{23} := \mathcal{B}_2 \odot \mathcal{B}_3$ denote the free amalgamated product of $\mathcal{B}_2$ and $\mathcal{B}_3$, and let $h_{B_i-B_{23}}$ $(i = 2,3)$ be the corresponding embeddings. Thus, we have

$$h_{A-B_2} \circ h_{B_2-B_{23}} \quad = \quad h_{A-B_3} \circ h_{B_3-B_{23}}. \tag{3.8}$$

Now, we consider $(\mathcal{A}, \mathcal{B}_1, \mathcal{B}_{23})$ with the embeddings $h_{A-B_1} : \mathcal{A} \to \mathcal{B}_1$ and $h_{A-B_2} \circ h_{B_2-B_{23}} : \mathcal{A} \to \mathcal{B}_{23}$ as amalgamation base. Let $\mathcal{B}_{123} := \mathcal{B}_1 \odot \mathcal{B}_{23}$ be the corresponding free amalgamated product with embeddings $h_{B_1-B_{123}}$ and $h_{B_{23}-B_{123}}$. By definition of the amalgamated product, these embeddings satisfy

$$h_{A-B_1} \circ h_{B_1-B_{123}} \quad = \quad (h_{A-B_2} \circ h_{B_2-B_{23}}) \circ h_{B_{23}-B_{123}}. \tag{3.9}$$

We show that $\mathcal{B}_{123}$ closes the simultaneous amalgamation base $(\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$. To this purpose, we define

$$h_{B_i-B_{123}} \quad := \quad h_{B_i-B_{23}} \circ h_{B_{23}-B_{123}} \; (i = 2,3). \tag{3.10}$$

It is easy to see that, with this definition, (3.8) and (3.9) imply

$$h_{A-B_1} \circ h_{B_1-B_{123}} = h_{A-B_2} \circ h_{B_2-B_{123}} = h_{A-B_3} \circ h_{B_3-B_{123}},$$

i.e., $\mathcal{B}_{123}$ indeed closes the simultaneous amalgamation base. Because of the assumption that $\mathcal{B}_1 \odot (\mathcal{B}_2 \odot \mathcal{B}_3) \in Adm(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$, we know that $\mathcal{B}_{123} \in Adm(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$. Thus, it remains to be shown that the admissible simultaneous amalgamated product $\mathcal{B}_{123}$ is in fact free.

Assume that $\mathcal{D} \in Adm(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$ is an admissible simultaneous amalgamated product with embeddings $g_{B_i-D} : \mathcal{B}_i \to \mathcal{D}$ $(i = 1,2,3)$, which thus satisfy

$$h_{A-B_1} \circ g_{B_1-D} = h_{A-B_2} \circ g_{B_2-D} = h_{A-B_3} \circ g_{B_3-D}. \tag{3.11}$$

Equation (3.11), together with our assumption that the classes of admissible structures satisfy $Adm(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3) \subseteq Adm(\mathcal{B}_2, \mathcal{B}_3)$, implies that $\mathcal{D}$ is also an admissible amalgamated product of $\mathcal{B}_2$ and $\mathcal{B}_3$. Since $\mathcal{B}_{23}$ is the free amalgamated product of $\mathcal{B}_2$ and $\mathcal{B}_3$, there exists a unique homomorphism $f_{B_{23}-D} : \mathcal{B}_{23} \to \mathcal{D}$ such that

$$g_{B_i-D} \quad = \quad h_{B_i-B_{23}} \circ f_{B_{23}-D} \ (i = 2, 3). \tag{3.12}$$

Because of our assumption $Adm(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3) \subseteq Adm(\mathcal{B}_1, \mathcal{B}_2 \odot \mathcal{B}_3)$, we know that $\mathcal{D} \in Adm(\mathcal{B}_1, \mathcal{B}_2 \odot \mathcal{B}_3)$. In addition, we have $h_{A-B_1} \circ g_{B_1-D} = h_{A-B_2} \circ g_{B_2-D} = h_{A-B_2} \circ h_{B_2-B_{23}} \circ f_{B_{23}-D}$ (the first identity holds because of (3.11) and the second because of (3.12)). This shows that $\mathcal{D}$ with the embeddings g_{B_1-D} and $f_{B_{23}-D}$ is an admissible amalgamated product of $\mathcal{B}_1$ and $\mathcal{B}_{23}$. Since $\mathcal{B}_{123}$ is the free amalgamated product of $\mathcal{B}_1$ and $\mathcal{B}_{23}$, there exists a unique homomorphism $f_{B_{123}-D} : \mathcal{B}_{123} \to \mathcal{D}$ such that

$$g_{B_1-D} \quad = \quad h_{B_1-B_{123}} \circ f_{B_{123}-D}, \tag{3.13}$$

$$f_{B_{23}-D} \quad = \quad h_{B_{23}-B_{123}} \circ f_{B_{123}-D}. \tag{3.14}$$

We must show that $g_{B_i-D} = h_{B_i-B_{123}} \circ f_{B_{123}-D}$ for $i = 1, 2, 3$. For $i = 1$, this is just identity (3.13). For $i = 2, 3$, we have $h_{B_i-B_{123}} \circ f_{B_{123}-D} = h_{B_i-B_{23}} \circ h_{B_{23}-B_{123}} \circ f_{B_{123}-D} = h_{B_i-B_{23}} \circ f_{B_{23}-D} = g_{B_i-D}$ (the first identity holds by (3.10), the second by (3.14), and the third by (3.12)).

It remains to be shown that $f_{B_{123}-D}$ is unique with this property. Thus, assume that $e_{B_{123}-D} : \mathcal{B}_{123} \to \mathcal{D}$ is a homomorphism satisfying

$$g_{B_i-D} \quad = \quad h_{B_i-B_{123}} \circ e_{B_{123}-D} \ (i = 1, 2, 3). \tag{3.15}$$

The identity (3.15) together with (3.10) yields

$$g_{B_i-D} = h_{B_i-B_{23}} \circ h_{B_{23}-B_{123}} \circ e_{B_{123}-D} \ (i = 2, 3).$$

Since $f_{B_{23}-D}$ is the unique morphism satisfying (3.12), this implies

$$f_{B_{23}-D} \quad = \quad h_{B_{23}-B_{123}} \circ e_{B_{123}-D}. \tag{3.16}$$

Now, consider (3.15) for $i = 1$ and (3.16): Since $f_{B_{123}-D}$ is the unique homomorphism satisfying (3.13) and (3.14), these two identities imply $f_{B_{123}-D} = e_{B_{123}-D}$. ☐

Obviously, a dual lemma holds for $(\mathcal{B}_1 \odot \mathcal{B}_2) \odot \mathcal{B}_3$. Since the free simultaneous amalgamated product is unique, this implies the next theorem.

Theorem 3.17 (Associativity of free amalgamation)
Let $\Gamma \subseteq \Sigma_1 \cap \Sigma_2 \cap \Sigma_3$, and let $\mathcal{A}^\Gamma, \mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2}, \mathcal{B}_3^{\Sigma_3}$ be structures with fixed homomorphic embeddings $h^\Gamma_{A-B_1} : \mathcal{A}^\Gamma \to \mathcal{B}_1^{\Sigma_1}$, $h^\Gamma_{A-B_2} : \mathcal{A}^\Gamma \to \mathcal{B}_2^{\Sigma_2}$, and $h^\Gamma_{A-B_3} : \mathcal{A}^\Gamma \to \mathcal{B}_3^{\Sigma_3}$. Assume that the free amalgamated products $\mathcal{B}_2 \odot \mathcal{B}_3$, $\mathcal{B}_1 \odot (\mathcal{B}_2 \odot \mathcal{B}_3)$, $\mathcal{B}_1 \odot B_2$, and $(\mathcal{B}_1 \odot B_2) \odot \mathcal{B}_3$ exist, and that the classes of admissible structures satisfy

$$\begin{aligned}\{\mathcal{B}_1 \odot (\mathcal{B}_2 \odot \mathcal{B}_3), (\mathcal{B}_1 \odot B_2) \odot \mathcal{B}_3\} &\subseteq Adm(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3), \text{ and} \\ Adm(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3) &\subseteq Adm(\mathcal{B}_1, \mathcal{B}_2) \cap Adm(\mathcal{B}_1 \odot \mathcal{B}_2, \mathcal{B}_3) \cap \\ &\quad\; Adm(\mathcal{B}_2, \mathcal{B}_3) \cap Adm(\mathcal{B}_1, \mathcal{B}_2 \odot \mathcal{B}_3).\end{aligned}$$

Then we have $(\mathcal{B}_1 \odot \mathcal{B}_2) \odot \mathcal{B}_3 \simeq \mathcal{B}_1 \odot (\mathcal{B}_2 \odot \mathcal{B}_3)$, and this structure is the free simultaneous amalgamated product of $\mathcal{B}_1$, B_2, and $\mathcal{B}_3$ over $\mathcal{A}^\Gamma$.

4 Simply Combinable Structures

In this section we shall introduce the concept of a simply combinable (SC-) structure. This purely algebraic notion yields a large class of structures for which an amalgamated product can be obtained by an explicit construction, provided that the component structures have disjoint signatures. Quotient term algebras, but also other typical domains for constraint based reasoning such as the algebra of rational trees and (certain types of) feature structures belong to this class. Quotient term algebras will serve as motivating example for the abstract definitions. The need for using more general notions will be illustrated with the help of the algebra of rational trees [Col84, Mah88] and feature structures [APS94, SmT94].

4.1 Stable hulls and atom sets

Let E be an equational theory and V be a countably infinite set (of variables). The quotient algebra $\mathcal{T} := \mathcal{T}(\Sigma_F, V)/=_E$ is the free algebra over V for the class of all models of E. In particular, this means that this algebra is generated by V, and that every mapping from V into its carrier can be extended to an endomorphism of $\mathcal{T}(\Sigma_F, V)/=_E$. For every element $[t]$ of this algebra, there exists a finite subset $U \subseteq V$ such that $[t]$ is "generated by U," i.e., $[t]$ is in the subalgebra $\mathcal{T}(\Sigma_F, U)/=_E$ of $\mathcal{T}(\Sigma_F, V)/=_E$. Obviously, if $[t]$ is generated by U, then two homomorphisms that coincide on U also coincide on $[t]$.

When defining SC-structures we shall keep most of these properties. In particular, every SC-structure will have a distinguished subset of "atoms", and these atoms almost behave like variables of a quotient term algebra. However, we shall *not* demand that the underlying algebra of an SC-structure is generated by its atom set. Consider, as an example, the algebra of rational trees where leaves are labeled by constants or variables. This algebra is not generated by the set of variables (since "generated by" talks about a finite process whereas rational trees may be infinite). Still, two endomorphisms of this algebra that coincide on a set U of variables coincide on all trees that are built over U. This motivates the definition of stable hulls and atom sets given below.

Definition 4.1 *Let A_0, A_1 be subsets of the Σ-structure $\mathcal{A}^\Sigma$, and let $\mathcal{M} \leq End_{\mathcal{A}}^\Sigma$. Then A_0* stabilizes A_1 with respect to $\mathcal{M}$ *iff all elements h_1 and h_2 of $\mathcal{M}$ that coincide on A_0 also coincide on A_1. If $\mathcal{M} = End_{\mathcal{A}}^\Sigma$, then we say that A_0* strictly *stabilizes A_1.*

The reason for considering submonoids of $End_{\mathcal{A}}^\Sigma$ is that in some cases (such as for feature structures) not all endomorphisms will be of interest in our context. In the sequel, we consider a fixed Σ-structure $\mathcal{A}^\Sigma$; $\mathcal{M}$ always denotes a submonoid of $End_{\mathcal{A}}^\Sigma$.

Definition 4.2 *For $A_0 \subseteq A$ the* stable hull of A_0 with respect to $\mathcal{M}$ *is the set*

$$SH_{\mathcal{M}}^{\mathcal{A}}(A_0) := \{a \in A;\ A_0 \text{ stabilizes } \{a\} \text{ with respect to } \mathcal{M}\}.$$

The following two lemmas show that the stable hull of a set A_0 has properties that are similar to those of the subalgebra generated by A_0. Note, however, that the stable hull can be larger than the generated subalgebra (see Example 4.9).

Lemma 4.3 *Let A_0 be a subset of the carrier A of $\mathcal{A}^\Sigma$ such that $SH_{\mathcal{M}}^{\mathcal{A}}(A_0) \neq \emptyset$. Then $SH_{\mathcal{M}}^{\mathcal{A}}(A_0)$ is a Σ-substructure of $\mathcal{A}^\Sigma$, and $A_0 \subseteq SH_{\mathcal{M}}^{\mathcal{A}}(A_0)$.*

Proof. Obviously, $A_0 \subseteq SH_{\mathcal{M}}^{\mathcal{A}}(A_0)$. Let $f \in \Sigma$ be an n-ary function symbol, and let $a_1, \ldots, a_n$ be elements of $SH_{\mathcal{M}}^{\mathcal{A}}(A_0)$. We must show that $f_{\mathcal{A}}(a_1, \ldots, a_n) \in SH_{\mathcal{M}}^{\mathcal{A}}(A_0)$. Let h_1 and h_2 be two endomorphisms in $\mathcal{M}$ that coincide on A_0. By assumption, h_1 and h_2 coincide on $a_1, \ldots, a_n$. Thus $h_1(f_{\mathcal{A}}(a_1, \ldots, a_n)) = f_{\mathcal{A}}(h_1(a_1), \ldots, h_1(a_n)) = f_{\mathcal{A}}(h_2(a_1), \ldots, h_2(a_n)) = h_2(f_{\mathcal{A}}(a_1, \ldots, a_n))$. $\square$

Lemma 4.4 *Let A_0, A_1 be subsets of the Σ-structure $\mathcal{A}^\Sigma$, and let $h \in \mathcal{M}$. If $h(A_0) \subseteq SH^{\mathcal{A}}_{\mathcal{M}}(A_1)$, then $h(SH^{\mathcal{A}}_{\mathcal{M}}(A_0)) \subseteq SH^{\mathcal{A}}_{\mathcal{M}}(A_1)$.*

Proof. Suppose that $h(A_0) \subseteq SH^{\mathcal{A}}_{\mathcal{M}}(A_1)$. Let g_1 and g_2 be two endomorphisms in $\mathcal{M}$ that coincide on A_1. Then g_1 and g_2 coincide on $SH^{\mathcal{A}}_{\mathcal{M}}(A_1)$. Thus $h \circ g_1$ and $h \circ g_2$ coincide on A_0. It follows that $h \circ g_1$ and $h \circ g_2$ coincide on $SH^{\mathcal{A}}_{\mathcal{M}}(A_0)$, and g_1 and g_2 coincide on $h(SH^{\mathcal{A}}_{\mathcal{M}}(A_0))$. $\square$

Definition 4.5 *The set $X \subseteq A$ is an $\mathcal{M}$-*atom set for *$\mathcal{A}^\Sigma$ if every mapping $X \to A$ can be extended to an endomorphism in $\mathcal{M}$. If $\mathcal{M} = End^\Sigma_{\mathcal{A}}$, then X is simply called an* atom set *for $\mathcal{A}^\Sigma$.*

For $\mathcal{T}$, the set of variables V is an atom set. Two subalgebras generated by subsets V_0, V_1 of V of the same cardinality are isomorphic. The same holds for atom sets and their stable hulls.

Lemma 4.6 *Let X_0, X_1 be two non-empty $\mathcal{M}$-atom sets of $\mathcal{A}^\Sigma$ of the same cardinality. Then every bijection $h_0 : X_0 \to X_1$ can be extended to an isomorphism between $SH^{\mathcal{A}}_{\mathcal{M}}(X_0)$ and $SH^{\mathcal{A}}_{\mathcal{M}}(X_1)$.*

Proof. Let $h_0 : X_0 \to X_1$ be bijective, and let $h_1 : X_1 \to X_0$ denote the inverse mapping. Since X_0 and X_1 are $\mathcal{M}$-atom sets, both mappings can be extended to endomorphisms $\widehat{h}_0$ and $\widehat{h}_1$ in $\mathcal{M}$. Now $(\widehat{h}_0 \circ \widehat{h}_1) \in \mathcal{M}$ is an endomorphism that coincides with $Id_A \in \mathcal{M}$ on X_0. Therefore, it coincides with Id_A on $SH^{\mathcal{A}}_{\mathcal{M}}(X_0)$.

Let g_i denote the restriction of $\widehat{h}_i$ to $SH^{\mathcal{A}}_{\mathcal{M}}(X_i)$ $(i = 0, 1)$. The previous lemma shows that

$$\begin{aligned} g_0 : SH^{\mathcal{A}}_{\mathcal{M}}(X_0) &\to SH^{\mathcal{A}}_{\mathcal{M}}(X_1), \\ g_1 : SH^{\mathcal{A}}_{\mathcal{M}}(X_1) &\to SH^{\mathcal{A}}_{\mathcal{M}}(X_0). \end{aligned}$$

We have $g_0 \circ g_1 = Id_{\mathrm{SH}^{\mathcal{A}}_{\mathcal{M}}(X_0)}$, which implies that g_0 is injective and g_1 is surjective. Symmetrically, we can show that g_0 is surjective and g_1 is injective. Thus, g_0 and g_1 are bijective homomorphisms, and g_i is the inverse of g_{1-i} $(i = 0, 1)$. $\square$

Another important property of generators in free algebras that can be generalized to atom sets is given by the next lemma:

Lemma 4.7 *Let X be an infinite $\mathcal{M}$-atom set of the countably infinite Σ-structure $\mathcal{A}^{\Sigma}$, and let $X_0 \subset X$ be finite. Then every mapping $h_0 : X_0 \to A$ can be extended to a surjective endomorphism in $\mathcal{M}$.*

Proof. Obviously, h_0 can be extended to a surjective mapping $h_1 : X \to A$. Since X is an $\mathcal{M}$-atom set, h_1 can be extended to an endomorphism $h_2 \in \mathcal{M}$ of $\mathcal{A}^{\Sigma}$. By construction, h_2 is surjective. □

4.2 SC-structures—examples and basic properties

We are now ready to introduce the main concept of this paper.

Definition 4.8 *A countably infinite Σ-structure $\mathcal{A}^{\Sigma}$ is an* SC-structure *iff there exists a monoid $\mathcal{M} \leq End_{\mathcal{A}}^{\Sigma}$ such that $\mathcal{A}^{\Sigma}$ has an infinite $\mathcal{M}$-atom set X where every $a \in A$ is stabilized by a finite subset of X with respect to $\mathcal{M}$. We denote this SC-structure by $(\mathcal{A}^{\Sigma}, \mathcal{M}, X)$. If $\mathcal{M} = End_{\mathcal{A}}^{\Sigma}$, then $(\mathcal{A}^{\Sigma}, End_{\mathcal{A}}^{\Sigma}, X)$ is called a* strong SC-structure.

Examples 4.9 The following list of examples shows that in fact many solution domains for symbolic constraints are SC-structures.

(1) Let Σ_F be a finite set of function symbols. The free algebra $\mathcal{T}(\Sigma_F, V)/{=_E}$ modulo the equational theory E with countably infinite generator set V is a strong SC-structure with atom set V. The same holds for free structures, as considered in [BaS94a].

(2) Let K be a field, let $\Sigma_K := \{+\} \cup \{s_k; k \in K\}$. The K-vector space spanned by a countably infinite basis X is a strong SC-structure over the atom set X. Here "+" is interpreted as addition of vectors, and s_k denotes scalar multiplication with $k \in K$.

(3) Let Σ_F be a finite set of function symbols, and let $\mathcal{R}^{\Sigma_F}$ be the algebra of rational trees ([Col84, Mah88]) where leaves are labelled with constants from Σ_F or with variables from the countably infinite set (of variables) V. It is easy to see that every mapping $V \to R$ can be extended to a unique endomorphism of $\mathcal{R}^{\Sigma_F}$, and that $(\mathcal{R}^{\Sigma_F}, End_{\mathcal{R}}^{\Sigma_F}, V)$ is a strong SC-structure. Note, however, that $\mathcal{R}^{\Sigma_F}$ is not generated by V: it is only—and exactly—the subset of *finite* trees which is generated by V.

(4) Let $V_{\mathrm{hfs}}(Y)$ be the set of all nested, hereditarily finite (standard, i.e., wellfounded) sets over the countably infinite set of "urelements" Y. Thus, each $M \in V_{\mathrm{hfs}}(Y)$ is finite, and the elements of M are either in Y or in $V_{\mathrm{hfs}}(Y)$, the same holds for elements of elements etc. There are no infinite descending membership sequences. Since union is not defined for the urelements $y \in Y$, the urelements will not be treated as sets here. Let $X := \{\{y\} \mid y \in Y\}$. Let $h : X \to V_{\mathrm{hfs}}(Y)$ be an arbitrary mapping. We want to show that there exists a unique extension of h to a mapping $\hat{h} : V_{\mathrm{hfs}}(Y) \to V_{\mathrm{hfs}}(Y)$ that is homomorphic with respect a signature that contains a binary symbol for union "$\cup$", a unary symbol for set construction $\{\cdot\}$, and a constant ϵ that denotes the empty set. We have to define $h(\emptyset) := \emptyset$. Each non-empty $M \in V_{\mathrm{hfs}}(Y)$ can uniquely be represented in the form $M = x_1 \cup \ldots \cup x_k \cup \{M_1\} \cup \ldots \cup \{M_l\}$ where $x_i \in X$, for $1 \leq i \leq k$, and where the M_i are the elements of M that belong to $V_{\mathrm{hfs}}(Y)$. By induction (on nesting depth), we may assume that $\hat{h}(M_i)$ is already defined $(1 \leq i \leq l)$. Obviously $\hat{h}(M) := h(x_1) \cup \ldots \cup h(x_k) \cup \{\hat{h}(M_1)\} \cup \ldots \cup \{\hat{h}(M_l)\}$ is one and the only way of extending $\hat{h}$ in a homomorphic way to the set M of deeper nesting. For $M = x \in X$ we obtain $\hat{h}(x) = h(x)$, thus $\hat{h}$ is an extension of h. Moreover, each mapping $\hat{h}$ is in fact homomorphic with respect to the given signature. Thus $V_{\mathrm{hfs}}(Y)$, under the given signature, is a strong SC-structure with atom set X.

(5) Similarly it can be seen that the domain $V_{\mathrm{hfnws}}(Y)$ of heriditarily finite *non-wellfounded* sets[3] over a countably infinite set of urelements Y, under the same signature, is a strong SC-structure over the atom set $X = \{\{y\} \mid y \in Y\}$.

(6) The two domains $V_{\mathrm{hfl}}(Y)$ and $V_{\mathrm{hfnwl}}(Y)$ of nested, hereditarily finite (1) wellfounded or (2) non-wellfounded lists over the countably infinite set of urelements Y, under a signature with a binary symbol for concatenation "$\circ$", a (unary) symbol for list construction $\langle\cdot\rangle : l \mapsto \langle l \rangle$, and a constant *nil* for the empty list, are strong SC-structures over the atom set $X = \{\langle y \rangle; y \in Y\}$ of all lists with one element $y \in Y$. Formally, these domains can be described as the set of all (1) finite or (2) rational trees where the topmost node has label "$\langle\ \rangle$" (representing a list constructor of varying finite arity), nodes with successors have label "$\langle\ \rangle$", and leaves have labels $y \in Y$ or "$\langle\ \rangle$".

(7) Let *Lab*, *Fea*, and X be mutually disjoint infinite sets of labels, features, and atoms respectively. Following [APS94], we define a feature tree to be a

[3] Non-wellfounded sets, sometimes called hypersets, became prominent through [Acz88]. They can have infinite descending membership sequences. The heriditarily finite non-wellfounded sets are those having a "finite picture," see [Acz88] for details.

partial function $t : \mathit{Fea}^* \to \mathit{Lab} \cup X$ whose domain is prefix closed (i.e., if $pq \in \mathit{dom}(t)$ then $p \in \mathit{dom}(t)$ for all words $p, q \in \mathit{Fea}^*$), and in which atoms do not label interior nodes (i.e., if $p(t) = x \in X$ then there is no $f \in \mathit{Fea}$ with $pf \in \mathit{dom}(t)$). As usual, *rational* feature trees are required to have only finitely many subtrees. In addition, they must be finitely branching.

We use the set R of all rational feature trees as carrier set of a structure $\mathcal{R}^\Sigma$ whose signature contains a unary predicate L for every label $L \in \mathit{Lab}$, and a binary predicate f for every $f \in \mathit{Fea}$. The interpretation $L_\mathcal{R}$ of L in $\mathcal{R}$ is the set of all rational feature trees having root label L. The interpretation $f_\mathcal{R}$ of f consists of all pairs $(t_1, t_2) \in R \times R$ such that $t_1(f)$ is defined and t_2 is the subtree of t_1 at f. The structure $\mathcal{R}^\Sigma$ defined this way can be seen as a non-ground version of the solution domain used in [APS94].

Each mapping $h : X \to R$ has a unique extension to an endomorphism of $\mathcal{R}^\Sigma$ that acts like a substitution, replacing each leaf with label $x \in X$ by the feature tree $h(x)$. With composition, the set of these substitution-like endomorphisms yield a monoid $\mathcal{M}$. Thus $(\mathcal{R}^\Sigma, \mathcal{M}, X)$ is an SC-structure. We shall call it the non-ground structure of rational feature trees. In this case, we do not have a strong SC-structure since $\mathcal{R}^\Sigma$ has endomorphisms that modify non-leaf nodes (e.g., by introducing new feature-edges for such internal nodes).

Now suppose that we introduce, following [SmT94], additional arity predicates F for every finite set $F \subseteq \mathit{Fea}$. The interpretation $F_\mathcal{R}$ of F consists of all feature trees t where the root of t has a label $L \in \mathit{Lab}$ and where F is (exactly) the set of all features departing from the root of t. Let Δ be the extended signature. Then $(\mathcal{R}^\Delta, \mathcal{M}, X)$ is a strong SC-structure. We shall call it the non-ground structure of rational feature trees with arity.

As we may see from the previous examples, there is often a ground variant of a given SC-structure. The following definition formalizes this relationship.

Definition 4.10 *Let $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ be an SC-structure such that $SH^\mathcal{A}_\mathcal{M}(\emptyset)$ is non-empty. Then $\mathcal{A}^\Sigma_H := SH^\mathcal{A}_\mathcal{M}(\emptyset)$ is called the* ground substructure *of $(\mathcal{A}^\Sigma, \mathcal{M}, X)$.*

Before we can turn to the combination of SC-structures, we must establish some useful properties of these structures.

Lemma 4.11 *Let $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ be an SC-structure.*

1. *$\mathcal{A}^\Sigma = SH^\mathcal{A}_\mathcal{M}(X)$ and every mapping $X \to A$ has a* unique *extension to an endomorphism of $\mathcal{A}^\Sigma$ in $\mathcal{M}$.*

2. *Let $X_0 \subseteq X$. Then we have $SH^{\mathcal{A}}_{\mathcal{M}}(X_0) \cap X = X_0$.*

3. *For all finite sets $\{a_1, \ldots, a_n\} \subseteq A$ there exists a* unique minimal *finite subset Y of X such that $\{a_1, \ldots, a_n\} \subseteq SH^{\mathcal{A}}_{\mathcal{M}}(Y)$.*

Proof. (1) Since every element of A is stabilized by a finite subset of X, the $\mathcal{M}$-atom set X stabilizes the whole structure A with respect to $\mathcal{M}$, which means that $\mathcal{A}^{\Sigma} = SH^{\mathcal{A}}_{\mathcal{M}}(X)$. Existence of the extension in $\mathcal{M}$ follows from the fact that X is an $\mathcal{M}$-atom set, and uniqueness is an immediate consequence of $\mathcal{A}^{\Sigma} = SH^{\mathcal{A}}_{\mathcal{M}}(X)$.

(2) The inclusion $X_0 \subseteq SH^{\mathcal{A}}_{\mathcal{M}}(X_0)$ follows from Lemma 4.3. For the other direction, assume that an $\mathcal{M}$-atom $x \in X$ is in $SH^{\mathcal{A}}_{\mathcal{M}}(X_0) \setminus X_0$. Let $h_1, h_2 : X \to A$ be mappings that coincide on X_0, but differ on x. Because X is an $\mathcal{M}$-atom set, there are endomorphisms $\widehat{h}_1, \widehat{h}_2 \in \mathcal{M}$ extending h_1, h_2. Since $\widehat{h}_1$ and $\widehat{h}_2$ coincide on X_0, they coincide on $x \in SH^{\mathcal{A}}_{\mathcal{M}}(X_0)$. This is a contradiction to our assumption that h_1 and h_2 differ on x.

(3) Since $(\mathcal{A}^{\Sigma}, \mathcal{M}, X)$ is an SC-structure, every finite set $\{a_1, \ldots, a_n\} \subseteq A$ is stabilized by a finite subset of X with respect to $\mathcal{M}$. Let X_0, X_1 be two finite subsets of X such that $\{a_1, \ldots, a_n\} \subseteq SH^{\mathcal{A}}_{\mathcal{M}}(X_i)$ for $i = 0, 1$. We claim that $\{a_1, \ldots, a_n\} \subseteq SH^{\mathcal{A}}_{\mathcal{M}}(X_0 \cap X_1)$. In fact, let $h_0, h_1 \in \mathcal{M}$ be two endomorphisms that coincide on $X_0 \cap X_1$. We may choose an endomorphism $h_{0,1} \in \mathcal{M}$ that coincides with h_0 on X_0 and with h_1 on X_1. Such an endomorphism exists in $\mathcal{M}$ since $(\mathcal{A}^{\Sigma}, \mathcal{M}, X)$ is an SC-structure. Now h_0 and $h_{0,1}$ coincide on $\{a_1, \ldots, a_n\}$, and h_1 and $h_{0,1}$ coincide on $\{a_1, \ldots, a_n\}$. This shows that h_0 and h_1 coincide on $\{a_1, \ldots, a_n\}$, and thus we have proved $\{a_1, \ldots, a_n\} \subseteq SH^{\mathcal{A}}_{\mathcal{M}}(X_0 \cap X_1)$. Obviously, this implies that there exists a unique minimal finite subset Y of X such that $\{a_1, \ldots, a_n\} \subseteq SH^{\mathcal{A}}_{\mathcal{M}}(Y)$. □

The third statement of the lemma shows that the notion "is stabilized by" behaves better than the notion "is generated by." In fact, minimal sets of generators need not be unique, as demonstrated by the next example.

Example 4.12 We consider the quotient term algebra $\mathcal{T}(\Sigma_F, V)/\!=_E$, where Σ_F consists of one unary function symbol f, V is countably infinite, and $E = \{f(x) = f(y)\}$. Obviously, the carrier of $\mathcal{T}(\Sigma_F, V)/\!=_E$ consists of the $=_E$-classes $\{x_i\}$ for $x_i \in V$ and one additional class $[f(\cdot)] := \{f(t) \mid t \in T(\Sigma_F, V)\}$.

It is easy to see that for all $x_i \in V$, the element $[f(\cdot)]$ of $\mathcal{T}(\Sigma_F, V)/\!=_E$ is generated by $\{x_i\}$. However, $[f(\cdot)]$ is not generated by $\emptyset$. Thus, there are infinitely many minimal sets of generators of $[f(\cdot)]$.

Definition 4.13 *Let $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ be an SC-structure, and let $\{a_1, \ldots, a_n\} \subseteq A$. The* stabilizer $Stab_{\mathcal{M}}(a_1, \ldots, a_n)$ *of $\{a_1, \ldots, a_n\}$ is the (unique) minimal finite subset Y of X such that $\{a_1, \ldots, a_n\} \subseteq SH^{\mathcal{A}}_{\mathcal{M}}(Y)$.*

Using this notion of stabilizers, the validity of positive formulae in SC-structure can be characterized in an algebraic way. This characterization is essential for proving correctness of our method of combining constraint solvers for SC-structures.

Lemma 4.14 *Let $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ be an SC-structure, and let*

$$\gamma = \forall \vec{u}_1 \exists \vec{v}_1 \ldots \forall \vec{u}_k \exists \vec{v}_k \; \varphi(\vec{u}_1, \vec{v}_1, \ldots, \vec{u}_k, \vec{v}_k)$$

be a positive Σ-sentence. Then the following conditions are equivalent:

1. $\mathcal{A}^\Sigma \models \forall \vec{u}_1 \exists \vec{v}_1 \ldots \forall \vec{u}_k \exists \vec{v}_k \; \varphi(\vec{u}_1, \vec{v}_1, \ldots, \vec{u}_k, \vec{v}_k)$,

2. *there exist $\vec{x}_1 \in \vec{X}, \vec{e}_1 \in \vec{A}, \ldots, \vec{x}_k \in \vec{X}, \vec{e}_k \in \vec{A}$ such that*

(a) $\mathcal{A}^\Sigma \models \varphi(\vec{x}_1, \vec{e}_1, \ldots, \vec{x}_k, \vec{e}_k)$,

(b) *all $\mathcal{M}$-atoms in the sequences $\vec{x}_1, \ldots, \vec{x}_k$ are distinct,*

(c) *for all $j, 1 \leq j \leq k$, the components of $\vec{x}_j$ are not contained in $Stab_{\mathcal{M}}(\vec{e}_1) \cup \ldots \cup Stab_{\mathcal{M}}(\vec{e}_{j-1})$.*

Proof. "1 $\Rightarrow$ 2". First, select an arbitrary sequence $\vec{x}_1$ of distinct $\mathcal{M}$-atoms from X such that this tuple has the same length as $\vec{u}_1$. Since $\mathcal{A}^\Sigma$ satisfies γ, there exists a sequence $\vec{e}_1 \in \vec{A}$ such that

$$(*) \quad \mathcal{A}^\Sigma \models \forall \vec{u}_2 \exists \vec{v}_2 \ldots \forall \vec{u}_k \exists \vec{v}_k \; \varphi(\vec{x}_1, \vec{e}_1, \vec{u}_2, \vec{v}_2, \ldots, \vec{u}_k, \vec{v}_k).$$

Now, we may choose a finite sequence $\vec{x}_2$ of distinct $\mathcal{M}$-atoms from X such that this sequence has the same length as $\vec{u}_2$, and none of its components occurs in $Stab_{\mathcal{M}}(\vec{e}_1)$ or $\vec{x}_1$. This is possible because X is infinite by assumption, and $Stab_{\mathcal{M}}(\vec{e}_1)$ is finite.

Because of $(*)$, there exist a sequence $\vec{e}_2 \in \vec{A}$ such that

$$\mathcal{A}^\Sigma \models \forall \vec{u}_3 \exists \vec{v}_3 \ldots \forall \vec{u}_k \exists \vec{v}_k \; \varphi(\vec{x}_1, \vec{e}_1, \vec{x}_2, \vec{e}_2, \vec{u}_3, \vec{v}_3, \ldots, \vec{u}_k, \vec{v}_k).$$

Obviously, this argument can be iterated until Condition 2 of the lemma is proved.

"2 $\Rightarrow$ 1". Let $\vec{x}_1 \in \vec{X}, \vec{e}_1 \in \vec{A}, \ldots, \vec{x}_k \in \vec{X}, \vec{e}_k \in \vec{A}$ as in Condition 2 be given. We claim that this implies, for all $i, 0 \leq i \leq k$, the following condition C_i:

C_i: For all $\vec{a}_1 \in \vec{A}$ there exists $\vec{b}_1 \in \vec{A}$, ..., for all $\vec{a}_i \in \vec{A}$ there exists $\vec{b}_i \in \vec{A}$, and there exist $\vec{y}_{i+1}, \ldots, \vec{y}_k \in \vec{X}$, $\vec{b}_{i+1}, \ldots, \vec{b}_k \in \vec{A}$ such that

(a') $\mathcal{A}^\Sigma \models \varphi(\vec{a}_1, \vec{b}_1, \ldots, \vec{a}_i, \vec{b}_i, \vec{y}_{i+1}, \vec{b}_{i+1}, \ldots, \vec{y}_k, \vec{b}_k)$,

(b') all atoms occurring in the tuples $\vec{y}_{i+1}, \ldots, \vec{y}_k$ are distinct,

(c') for all $j, i < j \leq k$, no component of $\vec{y}_j$ occurs in $\bigcup_{\nu=1}^{j-1} Stab_{\mathcal{M}}(\vec{b}_\nu) \cup \bigcup_{\mu=1}^{i} Stab_{\mathcal{M}}(\vec{a}_\mu)$.

Obviously, the condition C_k is just Condition 1 of the lemma. We show that condition C_i holds for all $i, 0 \leq i \leq k$, by induction on i. For $i = 0$, validity of C_0 follows from Condition 2.

Now, assume that C_i holds for some $i, 0 \leq i < k$. To show C_{i+1}, assume that an arbitrary sequence $\vec{a}_{i+1} \in \vec{A}$ is given. For $j = i+1, \ldots, k$, we define a mapping h_j from a finite set of atoms X_j to A by induction on j.

For $j = i+1$, the set X_{i+1} consists of $Stab_{\mathcal{M}}(\vec{b}_{i+1}) \cup \bigcup_{\nu=1}^{i}(Stab_{\mathcal{M}}(\vec{a}_\nu) \cup Stab_{\mathcal{M}}(\vec{b}_\nu))$ and the components of $\vec{y}_{i+1}$. The mapping h_{i+1} leaves all elements of $\bigcup_{\nu=1}^{i}(Stab_{\mathcal{M}}(\vec{a}_\nu) \cup Stab_{\mathcal{M}}(\vec{b}_\nu))$ invariant. It maps (each component of) $\vec{y}_{i+1}$ to (the corresponding component of) $\vec{a}_{i+1}$. The elements of $Stab_{\mathcal{M}}(\vec{b}_{i+1})$ that have not yet obtained an image this way are mapped in an arbitrary way. Note that this definition of h_{i+1} is consistent because of (b') and (c') of C_i.

Now assume that X_j, h_j are already defined (for some $i+1 \leq j < k$). The set X_{j+1} is obtained as the union of X_j with $Stab_{\mathcal{M}}(\vec{b}_{j+1})$ and the components of $\vec{y}_{j+1}$. The mapping h_{j+1} is obtained as follows:

1. Its restriction to X_j coincides with h_j.

2. Let $\vec{z}_j$ be a tuple of distinct atoms such that no component of $\vec{z}_j$ occurs in $Stab_{\mathcal{M}}(h_j(X_j))$. (Such a tuple exists since the set of atoms was assumed to be infinite, and $Stab_{\mathcal{M}}(h_j(X_j))$ is finite.) The mapping h_{j+1} maps (each component of) $\vec{y}_{j+1}$ to (the corresponding component of) $\vec{z}_{j+1}$.

3. The elements of $Stab_{\mathcal{M}}(\vec{b}_{i+1})$ that have not yet obtained an image this way are mapped in an arbitrary way.

Note that Condition 1 does not conflict with Condition 2 since (b') and (c') of C_i imply that none of the components of $\vec{y}_{j+1}$ occurs in X_j.

Since X is an infinite $\mathcal{M}$-atom set of the countably infinite Σ-structure $\mathcal{A}^\Sigma$, and X_k is a finite subset of X, Lemma 4.7 implies that there exists a surjective endomorphism $H \in \mathcal{M}$ that extends h_k. By definition of h_k, we have $H(\vec{a}_1) = \vec{a}_1$, $H(\vec{b}_1) = \vec{b}_1$, ..., $H(\vec{a}_i) = \vec{a}_i$, $H(\vec{b}_i) = \vec{b}_i$, $H(\vec{y}_{i+1}) = \vec{a}_{i+1}$, and for $i + 1 < j \leq k$, $H(\vec{y}_j) = \vec{z}_j$. Thus, Lemma 2.1 implies

$$\mathcal{A}^\Sigma \models \varphi(\vec{a}_1, \vec{b}_1, \ldots, \vec{a}_i, \vec{b}_i, \vec{a}_{i+1}, H(\vec{b}_{i+1}), \vec{z}_{i+2}, H(\vec{b}_{i+2}), \ldots, \vec{z}_k, H(\vec{b}_k)).$$

This yields (a') of C_{i+1}. It is easy to see that the mapping h_k was constructed such that (b') and (c') hold as well. □

5 SC-Substructures and SC-Superstructures

In Section 6, where we describe how to construct amalgamated products of SC-structures, it will be helpful to embed a given SC-structure in a larger (isomorphic) SC-structure. For the case of term algebras modulo an equational theory this is trivial. In fact, if V_∞ is any countable superset of the countably infinite set V then $\mathcal{T}(\Sigma_F, V)/{=_E}$ is isomorphic to $\mathcal{T}(\Sigma_F, V_\infty)/{=_E}$. For SC-structures, a similar property holds, which is, however, harder to prove. For this reason, we treat this problem in a separate, rather technical section. The reader who is eager to see how amalgamated products can be constructed may skip this section, and—for the moment—just believe its results.

Let $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ be an SC-structure, let X_0 be an infinite subset of X, and let $\mathcal{A}_0^\Sigma := SH_{\mathcal{M}}^{\mathcal{A}}(X_0)$. Our first goal is to show that $\mathcal{A}_0^\Sigma$ is an SC-structure with atom set X_0, and that there are close connections between this SC-structure and the SC-structure $(\mathcal{A}^\Sigma, \mathcal{M}, X)$. This will justify to call $\mathcal{A}_0^\Sigma$ an isomorphic SC-substructure of $\mathcal{A}^\Sigma$.

Lemma 5.1 *There exists an isomorphism $h_{A-A_0} : \mathcal{A}^\Sigma \rightarrow \mathcal{A}_0^\Sigma$ that maps X bijectively to X_0.*

Proof. By Lemma 4.11, $\mathcal{A}^\Sigma = SH_{\mathcal{M}}^{\mathcal{A}}(X)$, and thus Lemma 4.6 implies that every bijection between X and X_0 can be extended to an isomorphism from $\mathcal{A}^\Sigma$ to $\mathcal{A}_0^\Sigma$. □

Let $h_{A_0-A} := h_{A-A_0}^{-1}$ be the inverse isomorphism. For $m \in End_{\mathcal{A}}^{\Sigma}$, the mapping $m_\downarrow := h_{A_0-A} \circ m \circ h_{A-A_0}$ is obviously an endomorphism of $\mathcal{A}_0^\Sigma$. We define $\mathcal{M}_0 := \{m_\downarrow \mid m \in \mathcal{M}\}$.

Lemma 5.2 1. *$\mathcal{M}_0$ is a submonoid of $End^{\Sigma}_{\mathcal{A}_0}$.*

2. *The mapping $H_{\downarrow} : m \mapsto m_{\downarrow}$ is an isomorphism between the monoids $End^{\Sigma}_{\mathcal{A}}$ and $End^{\Sigma}_{\mathcal{A}_0}$.*

3. *$\mathcal{M}_0 = End^{\Sigma}_{\mathcal{A}_0}$ if, and only if, $\mathcal{M} = End^{\Sigma}_{\mathcal{A}}$.*

Proof. (1) Since

$$\begin{aligned} m_{\downarrow} \circ m'_{\downarrow} &= h_{A_0-A} \circ m \circ h_{A-A_0} \circ h_{A_0-A} \circ m' \circ h_{A-A_0} \\ &= h_{A_0-A} \circ m \circ m' \circ h_{A-A_0} \\ &= (m \circ m')_{\downarrow}, \end{aligned}$$

$\mathcal{M}_0$ is a submonoid of $End^{\Sigma}_{\mathcal{A}_0}$, and $H_{\downarrow}$ is a homomorphism between the monoids $End^{\Sigma}_{\mathcal{A}}$ and $End^{\Sigma}_{\mathcal{A}_0}$.

(2) There is a dual homomorphism

$$H_{\uparrow} : End^{\Sigma}_{\mathcal{A}_0} \to End^{\Sigma}_{\mathcal{A}} : m \mapsto m_{\uparrow} := h_{A-A_0} \circ m \circ h_{A_0-A},$$

and it is easy to see that $H_{\downarrow} \circ H_{\uparrow}$ is the identity on $End^{\Sigma}_{\mathcal{A}}$, and $H_{\uparrow} \circ H_{\downarrow}$ is the identity on $End^{\Sigma}_{\mathcal{A}_0}$. Thus, both are isomorphisms that are inverse to each other.

(3) Since $H_{\downarrow}$ is bijective, the images $\mathcal{M}_0$ of $\mathcal{M}$ under $H_{\downarrow}$ is equal to $End^{\Sigma}_{\mathcal{A}_0}$ iff $\mathcal{M} = End^{\Sigma}_{\mathcal{A}}$. □

Lemma 5.3 1. *$(\mathcal{A}^{\Sigma}_0, \mathcal{M}_0, X_0)$ is an SC-structure.*

2. *$(\mathcal{A}^{\Sigma}_0, \mathcal{M}_0, X_0)$ is strong iff $(\mathcal{A}^{\Sigma}, \mathcal{M}, X)$ is strong.*

3. *Every mapping $g_{X_0-A} : X_0 \to A$ can be extended to a homomorphism $g_{A_0-A} : \mathcal{A}^{\Sigma}_0 \to \mathcal{A}^{\Sigma}$. If $(\mathcal{A}^{\Sigma}_0, \mathcal{M}_0, X_0)$ is strong, then this extension is unique.*

4. *Let X'_0 be such that $X_0 \subseteq X'_0 \subseteq X$. Every bijection $g_0 : X_0 \to X'_0$ can be extended to an isomorphism between $\mathcal{A}^{\Sigma}_0 = SH^{\mathcal{A}}_{\mathcal{M}}(X_0)$ and $SH^{\mathcal{A}}_{\mathcal{M}}(X'_0)$. If $(\mathcal{A}^{\Sigma}_0, \mathcal{M}_0, X_0)$ is strong, then this extension is unique.*

Proof. (1.1) First, we show that X_0 is an $\mathcal{M}_0$-atom set of $\mathcal{A}^{\Sigma}_0$. Let $g_{X_0-A_0} : X_0 \to A_0$ be a mapping. There is a corresponding mapping

$$g_{X-A} : X \to A : x \mapsto h_{A_0-A}(g_{X_0-A_0}(h_{A-A_0}(x))).^{4}$$

[4] Recall that h_{A-A_0} maps X to X_0.

Since $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ is an SC-structure, there exists an extension g_{A-A} of g_{X-A} to an endomorphism in $\mathcal{M}$. Its image $(g_{A-A})_\downarrow$ is an endomorphism in $\mathcal{M}_0$, and it is easy to see that this endomorphism extends $g_{X_0-A_0}$.

(1.2) Second, we show that every element a of A_0 is stabilized by the set $h_{A-A_0}(Stab_{\mathcal{M}}(h_{A_0-A}(a)))$. Let $m_\downarrow$ and $m'_\downarrow$ be two endomorphisms in $\mathcal{M}_0$ that coincide on $h_{A-A_0}(Stab_{\mathcal{M}}(h_{A_0-A}(a)))$. For $x \in Stab_{\mathcal{M}}(h_{A_0-A}(a))$ we have

$$\begin{array}{rcl} m(x) & = & h_{A_0-A}(m_\downarrow(h_{A-A_0}(x))) \\ & = & h_{A_0-A}(m'_\downarrow(h_{A-A_0}(x))) = m'(x), \end{array}$$

which shows that m and m' coincide on $Stab_{\mathcal{M}}(h_{A_0-A}(a))$. Thus m and m' coincide on $h_{A_0-A}(a)$. We obtain

$$\begin{array}{rcl} m_\downarrow(a) & = & h_{A-A_0}(m(h_{A_0-A}(a))) \\ & = & h_{A-A_0}(m'(h_{A_0-A}(a))) \\ & = & m'_\downarrow(a). \end{array}$$

(1.3) Since $Stab_{\mathcal{M}}(h_{A_0-A}(a))$ is a finite subset of X, we know that the set $h_{A-A_0}(Stab_{\mathcal{M}}(h_{A_0-A}(a)))$ is a finite subset of X_0. Thus, every element of A_0 is stabilized by a finite subset of X_0, which completes the proof that $(\mathcal{A}_0^\Sigma, \mathcal{M}_0, X_0)$ is an SC-structure.

(2) Obviously, the third statement in Lemma 5.2 implies that the SC-structure $(\mathcal{A}_0^\Sigma, \mathcal{M}_0, X_0)$ is strong iff $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ is strong.

(3) Let $g_{X_0-A} : X_0 \to A$ be a mapping. We choose an arbitrary extension $g_{X-A} : X \to A$ of g_{X_0-A}. Since X is an $\mathcal{M}$-atom set, g_{X-A} can be extended to an endomorphism $g_{A-A} : \mathcal{A}^\Sigma \to \mathcal{A}^\Sigma$ in $\mathcal{M}$. The restriction g_{A_0-A} of g_{A-A} to $\mathcal{A}_0 = SH^{\mathcal{A}}_{\mathcal{M}}(X_0)$ is a homomorphism between $\mathcal{A}_0^\Sigma$ and $\mathcal{A}^\Sigma$ that extends g_{X_0-A}.

If $(\mathcal{A}_0^\Sigma, \mathcal{M}_0, X_0)$ is strong, then $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ is also strong. Let $h_{X-X_0} : X \to X_0$ be a bijection, and let h_{A-A_0} be an extension of h_{X-X_0} to an isomorphism from $\mathcal{A}^\Sigma = SH^{\mathcal{A}}_{\mathcal{M}}(X)$ to $\mathcal{A}_0^\Sigma = SH^{\mathcal{A}}_{\mathcal{M}}(X_0)$ (see Lemma 4.6). For all homomorphisms $g' : \mathcal{A}_0^\Sigma \to \mathcal{A}^\Sigma$ that extend g_{X_0-A}, the composition $h_{A-A_0} \circ g'$ is an endomorphism of $\mathcal{A}^\Sigma$ that extends the mapping $h_{X-X_0} \circ g_{X_0-A} : X \to A$. Since $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ is strong, all these endomorphisms $h_{A-A_0} \circ g'$ coincide.[5] Because h_{A-A_0} is an isomorphism, this implies that all homomorphisms g' extending g_{X_0-A} coincide, which yields the desired uniqueness result.

[5]The assumption "$(\mathcal{A}^\Sigma, \mathcal{M}, X)$ strong" is necessary, since otherwise uniqueness only holds for elements of $\mathcal{M}$, and we could not be sure that all $g_{A-A_0} \circ g'$ belong to $\mathcal{M}$.

(4) Let $g_0 : X_0 \to X'_0$ be a bijection, where $X_0 \subseteq X'_0 \subseteq X$. By Lemma 4.6, g_0 can be extended to an isomorphism between $\mathcal{A}_0^\Sigma = SH_{\mathcal{M}}^{\mathcal{A}}(X_0)$ and $\mathcal{A}_0'^\Sigma := SH_{\mathcal{M}}^{\mathcal{A}}(X'_0)$.

Suppose that $(\mathcal{A}_0^\Sigma, \mathcal{M}_0, X_0)$ is strong. Then $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ is also strong. Let h_{X-X_0} and h_{A-A_0} be defined as in part (3) of the proof. For all homomorphisms $g'' : \mathcal{A}_0^\Sigma \to \mathcal{A}_0'^\Sigma$ that extend g_0, the composition $h_{A-A_0} \circ g''$ is an endomorphism of $\mathcal{A}^\Sigma$ that extends the mapping $h_{X-X_0} \circ g_0$. Since $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ is strong, all these endomorphisms $h_{A-A_0} \circ g''$ coincide, Because h_{A-A_0} is an isomorphism, this implies that all homomorphisms g'' extending g_0 coincide. $\square$

Until now, we have seen that any countably infinite subset X_0 of the atom set X of an SC-structure $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ is an atom set for an appropriate isomorphic SC-substructure $(\mathcal{A}_0^\Sigma, \mathcal{M}_0, X_0)$ of $(\mathcal{A}^\Sigma, \mathcal{M}, X)$. In the remainder of this section, we use this result to go in the other direction, i.e., we show that a given SC-structure $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ can be embedded into an isomorphic SC-superstructure.

Theorem 5.4 *Let $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ be an SC-structure. There exists an SC-structure $(\mathcal{A}_\infty^\Sigma, \mathcal{M}_\infty, X_\infty)$ such that:*

(a0) $\mathcal{A}^\Sigma$ and $\mathcal{A}_\infty^\Sigma$ are isomorphic.

(a1) $\mathcal{A}^\Sigma = SH_{\mathcal{M}_\infty}^{\mathcal{A}_\infty}(X)$, $X \subset X_\infty$, and $X_\infty \setminus X$ is infinite.

(a2) $(\mathcal{A}_\infty^\Sigma, \mathcal{M}_\infty, X_\infty)$ is strong iff $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ is strong.

(a3) Every mapping $X \to A_\infty$ can be extended to a homomorphisms $h_{A-A_\infty}^\Sigma : \mathcal{A}^\Sigma \to \mathcal{A}_\infty^\Sigma$. If $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ is a strong SC-structure, then this extension is unique.

(a4) For every X' such that $X \subseteq X' \subseteq X_\infty$, every bijection $g : X \to X'$ can be extended to an isomorphism between $SH_{\mathcal{M}_\infty}^{\mathcal{A}_\infty}(X)$ and $SH_{\mathcal{M}_\infty}^{\mathcal{A}_\infty}(X')$. If $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ is a strong SC-structure, then this extension is unique.

Proof. (1) In the first part of the proof, we define the structure $\mathcal{A}_\infty^\Sigma$ and show that is isomorphic to $\mathcal{A}^\Sigma$. Let X_0 be an infinite subset of X such that $X \setminus X_0$ is infinite, and let $(\mathcal{A}_0^\Sigma, \mathcal{M}_0, X_0) = SH_{\mathcal{M}}^{\mathcal{A}}(X_0)$ be the isomorphic SC-substructure satisfying the properties stated in Lemma 5.3. Let $h_{A_0-A} : \mathcal{A}_0^\Sigma \to \mathcal{A}^\Sigma$ be an isomorphism that extends a bijection between the atom sets X_0 and X.

As carrier of the SC-superstructure to be constructed, we take an arbitrary countably infinite superset A_∞ of A such that $A_\infty \setminus A$ is infinite. Let X_∞ be a subset of A_∞ such that

1. $X \subseteq X_\infty$ and $X_\infty \setminus X$ is infinite,
2. $X_\infty \cap A = X$,
3. the sets $A \setminus (A_0 \cup X)$ and $A_\infty \setminus (A \cup X_\infty)$ have the same cardinality.

We extend h_{A_0-A} to a bijection $h_{A-A_\infty} : A \to A_\infty$ such that $h_{A-A_\infty}(X) = X_\infty$. This is possible because of our choice of h_{A_0-A} and of X_∞. In fact, by Lemma 4.11, $A = A_0 \uplus (X \setminus X_0) \uplus (A \setminus (A_0 \cup X))$ is a partitioning of A, and our assumptions ensure that $A_\infty = A \uplus (X_\infty \setminus X) \uplus (A_\infty \setminus (A \cup X_\infty))$ is a partitioning of A_∞. In addition, both $X \setminus X_0$ and $X_\infty \setminus X$ are countably infinite, and $A \setminus (A_0 \cup X)$ and $A_\infty \setminus (A \cup X_\infty)$ have the same cardinality by assumption.

The bijection h_{A-A_∞} and its inverse $h_{A_\infty-A} := h^{-1}_{A-A_\infty}$ can be used to define a Σ-structure $\mathcal{A}^\Sigma_\infty$ on the carrier A_∞ as follows: Let $f \in \Sigma$ be an n-ary function symbol, and $a_1, \ldots, a_n \in A_\infty$. We define the interpretation of f in $\mathcal{A}^\Sigma_\infty$ by

$$f_{\mathcal{A}_\infty}(a_1, \ldots, a_n) := h_{A-A_\infty}(f_{\mathcal{A}}(h_{A_\infty-A}(a_1), \ldots, h_{A_\infty-A}(a_n))).$$

Let $p \in \Sigma$ be an m-ary predicate symbol, and $a_1, \ldots, a_m \in A_\infty$. We define the interpretation of p in $\mathcal{A}^\Sigma_\infty$ by

$$p_{\mathcal{A}_\infty}[a_1, \ldots, a_n] :\Longleftrightarrow p_{\mathcal{A}}[h_{A_\infty-A}(a_1), \ldots, h_{A_\infty-A}(a_n)].$$

Note that this definition is compatible with the given Σ-structure on $A \subset A_\infty$ since h_{A_0-A}, i.e., the restriction of h_{A-A_∞} to A_0, is a Σ-isomorphism. With this definition, the mapping h_{A-A_∞} becomes an isomorphism between the Σ-structures $\mathcal{A}^\Sigma_\infty$ and $\mathcal{A}^\Sigma$, and $h_{A_\infty-A}$ is the inverse isomorphism.

(2) In the second part of the proof, we define the monoid $\mathcal{M}_\infty$, show that $(\mathcal{A}^\Sigma_\infty, \mathcal{M}_\infty, X_\infty)$ is an SC-structure, and that (a2) holds. The submonoid $\mathcal{M}$ of $End^\Sigma_{\mathcal{A}}$ induces a corresponding submonoid $\mathcal{M}_\infty$ of $End^\Sigma_{\mathcal{A}_\infty}$ as follows: For each $m \in End^\Sigma_{\mathcal{A}}$ we may define a corresponding endomorphism $m_\infty : a \mapsto m_\infty(a) := h_{A-A_\infty}(m(h_{A_\infty-A}(a)))$ of $\mathcal{A}_\infty$. Let $\mathcal{M}_\infty$ be the set $\{m_\infty \mid m \in \mathcal{M}\}$. Since

$$\begin{aligned} m_\infty \circ m'_\infty(a) &= h_{A-A_\infty}(m'(h_{A_\infty-A}(h_{A-A_\infty}(m(h_{A_\infty-A}(a)))))) \\ &= h_{A-A_\infty}(m \circ m'(h_{A_\infty-A}(a))) \\ &= (m \circ m')_\infty(a), \end{aligned}$$

$\mathcal{M}_\infty$ is in fact a submonoid of $End^\Sigma_{\mathcal{A}_\infty}$. As in the proof of Lemma 5.3, we can show that the mapping $m \mapsto m_\infty$ is an isomorphism between the monoids $End^\Sigma_{\mathcal{A}}$ and $End^\Sigma_{\mathcal{A}_\infty}$. In particular, this implies that $\mathcal{M}_\infty = End^\Sigma_{\mathcal{A}_\infty}$ if, and only if, $\mathcal{M} = End^\Sigma_{\mathcal{A}}$. Again, this will imply (a2) as soon as we have proved that $(\mathcal{A}^\Sigma_\infty, \mathcal{M}_\infty, X_\infty)$ is an SC-structure.

To this purpose, we show that X_∞ is an $\mathcal{M}_\infty$-atom set of $\mathcal{A}^\Sigma_\infty$. Let $g_{X_\infty - A_\infty} : X_\infty \to A_\infty$ be a mapping. There is a corresponding mapping

$$g_{X-A} : X \to A : x \mapsto h_{A_\infty - A}(g_{X_\infty - A_\infty}(h_{A-A_\infty}(x))).$$

Since $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ is an SC-structure, there exists an extension g_{A-A} of g_{X-A} to an endomorphism in $\mathcal{M}$. Its image $(g_{A-A})_\infty$ is an endomorphisms in $\mathcal{M}_\infty$, and it is easy to see that this endomorphism extends $g_{X_\infty - A_\infty}$. Thus, X_∞ is in fact an $\mathcal{M}_\infty$-atom set of of $\mathcal{A}^\Sigma_\infty$.

For a given $a \in A_\infty$ is also straightforward to verify that the finite set $h_{A-A_\infty}(Stab_{\mathcal{M}}(h_{A_\infty - A}(a)) \subset X_\infty$ stabilizes a with respect to $\mathcal{M}_\infty$. Thus we have shown that $(\mathcal{A}^\Sigma_\infty, \mathcal{M}_\infty, X_\infty)$ is an SC-structure. As mentioned before, (a2) holds.

(3) In order to prove (a1), it remains to be shown that $\mathcal{A}^\Sigma = SH^{\mathcal{A}_\infty}_{\mathcal{M}_\infty}(X)$. We know that $\mathcal{A}^\Sigma_0 = SH^{\mathcal{A}}_{\mathcal{M}}(X_0)$.

First, assume that $a \in A$. Since h_{A-A_∞} maps A_0 bijectively onto A, there exists $a_0 \in A_0$ such that $a = h_{A-A_\infty}(a_0)$. Now assume that m_∞ and m'_∞ coincide on X. It follows that m, m' coincide on X_0. In fact, let $x_0 \in X_0$. Then $h_{A-A_\infty}(x_0) \in X$, and thus

$$\begin{aligned} m(x_0) &= h_{A_\infty - A}(m_\infty(h_{A-A_\infty}(x_0))) \\ &= h_{A_\infty - A}(m'_\infty(h_{A-A_\infty}(x_0))) \\ &= m'(x_0). \end{aligned}$$

Thus, we know that m, m' coincide on $\mathcal{A}^\Sigma_0 = SH^{\mathcal{A}}_{\mathcal{M}}(X_0)$. It follows that

$$\begin{aligned} m_\infty(a) &= h_{A-A_\infty}(m(h_{A_\infty - A}(a))) \\ &= h_{A-A_\infty}(m(a_0)) \\ &= h_{A-A_\infty}(m'(a_0)) \\ &= h_{A-A_\infty}(m'(h_{A_\infty - A}(a))) \\ &= m'_\infty(a), \end{aligned}$$

and thus we have proved $a \in SH^{\mathcal{A}_\infty}_{\mathcal{M}_\infty}(X)$.

Second, assume that $a \in SH^{\mathcal{A}_\infty}_{\mathcal{M}_\infty}(X)$. We show that this implies that its image $h_{A_\infty - A}(a) \in SH^{\mathcal{A}}_{\mathcal{M}}(X_0) = \mathcal{A}^\Sigma_0$. Since the restriction of h_{A-A_∞} to A_0

maps A_0 onto A, it follows that $a = h_{A-A_\infty}(h_{A_\infty - A}(a)) \in A$. Thus, assume that $m, m' \in \mathcal{M}$ coincide on X_0. It is easy to see that this implies that m_∞, m'_∞ coincide on X, and thus they coincide on $a \in SH^{\mathcal{A}_\infty}_{\mathcal{M}_\infty}(X)$. It follows that

$$\begin{aligned} m(h_{A_\infty - A}(a)) &= h_{A_\infty - A}(m_\infty(a)) \\ &= h_{A_\infty - A}(m'_\infty(a)) \\ &= m'(h_{A_\infty - A}(a)), \end{aligned}$$

which proves $h_{A_\infty - A}(a) \in SH^{\mathcal{A}}_{\mathcal{M}}(X_0)$.

(4) In order to prove (a3), assume that $g_{X-A_\infty} : X \to A_\infty$ is an arbitrary mapping. There is a corresponding mapping

$$g_{X_0 - A} : X_0 \to A : x \mapsto h_{A_\infty - A}(g_{X-A_\infty}(h_{A-A_\infty}(x))).$$

By Lemma 5.3, g_{X_0-A} can be extended to a homomorphism $g^\Sigma_{A_0-A} : \mathcal{A}^\Sigma_0 \to \mathcal{A}^\Sigma$. Now

$$g^\Sigma_{A-A_\infty} : \mathcal{A}^\Sigma \to \mathcal{A}^\Sigma_\infty : a \mapsto h_{A-A_\infty}(g_{A_0-A}(h_{A_\infty-A}(a)))$$

is a homomorphism that extends g_{X-A_∞}. It is easy to see that there is a 1–1 correspondence between the extensions of g_{X-A_∞} to homomorphisms $\mathcal{A}^\Sigma \to \mathcal{A}^\Sigma_\infty$ and the extensions of g_{X_0-A} to homomorphisms $\mathcal{A}^\Sigma_0 \to \mathcal{A}^\Sigma$. Thus, in the case of strong SC-structures, uniqueness of the extension g_{A_0-A} of g_{X_0-A} implies uniqueness of the extension g_{A-A_∞} of g_{X-A_∞}.

(5) In order to prove (a4), assume that X' is a set with $X \subseteq X' \subseteq X_\infty$. Let $X'_0 := h_{A_\infty-A}(X')$. It is easy to check that $a \in A_\infty$ is stabilized by X' with respect to $\mathcal{M}_\infty$ if, and only if, $h_{A_\infty-A}(a) \in A$ is stabilized by X'_0 with respect to $\mathcal{M}$. Thus $h_{A-A_\infty}(SH^{\mathcal{A}}_{\mathcal{M}}(X'_0)) = SH^{\mathcal{A}_\infty}_{\mathcal{M}_\infty}(X')$. Let $g : X \to X'$ be a bijection, and define $g_0 : X_0 \to X'_0$ by $g_0(x_0) := h_{A_\infty-A}(g(h_{A-A_\infty}(x_0)))$. It is easy to see that there is a 1–1 correspondence between the extensions of g to isomorphisms $SH^{\mathcal{A}_\infty}_{\mathcal{M}_\infty}(X) \to SH^{\mathcal{A}_\infty}_{\mathcal{M}_\infty}(X')$ and the extensions of g_0 to isomorphisms $SH^{\mathcal{A}}_{\mathcal{M}}(X_0) \to SH^{\mathcal{A}}_{\mathcal{M}}(X'_0)$. Thus, (a4) follows from (4) of Lemma 5.3. □

6 Amalgamation of Simply Combinable Structures

Our motivation for introducing the class of SC-structures was, on the one hand, that it comprises many solution structures for interesting constraint languages. On the other hand, SC-structures over disjoint signatures allow for an explicit construction that closes any amalgamation base, as we shall see below. For two strong SC-structures over disjoint signatures, this construction yields the free amalgamated product of these structures. In the general case, the resulting structure also seems to play a unique role, but a precise characterization of this intuition has not yet been obtained. The following construction is almost identical to the amalgamation construction given in [BaS94a] for the case of free structures. There is just one essential difference. In [BaS94a], substructures that are generated by increasing sets of free generators are used in each step of the construction. Here, in the case of SC-structures, stable hulls (as defined in Definition 4.2) of increasing sets of atoms must be used instead.

6.1 The amalgamation construction

Let $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ and $(\mathcal{B}^\Delta, \mathcal{N}, X)$ be two SC-structures over disjoint signatures Σ and Δ. We consider the amalgamation base $(X, \mathcal{A}^\Sigma, \mathcal{B}^\Delta)$, where the common part is just the set of atoms X. Thus, the embedding "homomorphisms" $h_{X-A} : X \to A^\Sigma$ and $h_{X-B} : X \to B^\Delta$ are given by Id_X. In order to close this amalgamation base, we shall first embed $\mathcal{A}^\Sigma$ and $\mathcal{B}^\Delta$ into isomorphic superstructures. Let $(\mathcal{A}^\Sigma_\infty, \mathcal{M}_\infty, X_\infty)$ be an SC-superstructure of $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ satisfying (a0)–(a4) of Theorem 5.4. Analogously, there exists an SC-superstructure $(\mathcal{B}^\Delta_\infty, \mathcal{N}_\infty, Y_\infty)$ of $(\mathcal{B}^\Delta, \mathcal{N}, X)$ such that the corresponding properties (b0)–(b4) hold.

Starting from $\mathcal{A}^\Sigma_0 := \mathcal{A}^\Sigma$ and $\mathcal{B}^\Delta_0 := \mathcal{B}^\Delta$, we shall make a zig-zag construction that defines an ascending tower of Σ-structures $\mathcal{A}^\Sigma_n$, and similarly an ascending tower of Δ-structures $\mathcal{B}^\Delta_n$. These structures are connected by bijective mappings h_n and g_n. The amalgamated product is obtained as the limit structure, which obtains its functional and relational structure from both towers by means of the limits of the mappings h_n and g_n. Let $X_0 := Y_0 := X$.

$n = 0$: Consider $\mathcal{A}^\Sigma_0 = \mathcal{A}^\Sigma = SH^{\mathcal{A}_\infty}_{\mathcal{M}_\infty}(X_0)^\Sigma$. We interpret the "new" elements in $A_0 \setminus X_0$ as atoms in $\mathcal{B}^\Delta_\infty$. For this purpose, select a subset $Y_1 \subseteq Y_\infty$ such that $Y_1 \cap Y_0 = \emptyset$, $|Y_1| = |A_0 \setminus X_0|$, and the remaining complement $Y_\infty \setminus (Y_0 \cup Y_1)$ is countably infinite. Choose any bijection $h_0 : Y_0 \cup Y_1 \to A_0$ where $h_0|_{Y_0} = Id_{Y_0}$.

Consider $\mathcal{B}^\Delta_0 = \mathcal{B}^\Delta = SH^{\mathcal{B}_\infty}_{\mathcal{N}_\infty}(Y_0)^\Delta$. As for A_0, we interpret the "new" elements in $B_0 \setminus Y_0$ as atoms in $\mathcal{A}_\infty$. Select a subset $X_1 \subseteq X_\infty$ such that $X_1 \cap X_0 = \emptyset$, $|X_1| = |B_0 \setminus Y_0|$ and the remaining complement $X_\infty \setminus (X_0 \cup X_1)$ is countably infinite. Choose any bijection $g_0 : X_0 \cup X_1 \to B_0$ where $g_0|_{X_0} =$

Id_{X_0}.

$n \to n+1$: Suppose that the structures $\mathcal{A}_n^{\Sigma} = SH_{\mathcal{M}_\infty}^{\mathcal{A}_\infty}(\bigcup_{i=0}^n X_i)^{\Sigma}$ and $\mathcal{B}_n^{\Delta} = SH_{\mathcal{N}_\infty}^{\mathcal{B}_\infty}(\bigcup_{i=0}^n Y_i)^{\Delta}$ and the atom sets $X_{n+1} \subset (X_\infty \setminus \bigcup_{i=0}^n X_i)$ and $Y_{n+1} \subset (Y_\infty \setminus \bigcup_{i=0}^n Y_i)$ are already defined. We assume that the complements $X_\infty \setminus \bigcup_{i=0}^{n+1} X_i$ and $Y_\infty \setminus \bigcup_{i=0}^{n+1} Y_i$ are infinite. In addition, we assume that bijections

$$
\begin{array}{lll}
h_n: & B_{n-1} \cup Y_n \cup Y_{n+1} & \to A_n \\
g_n: & A_{n-1} \cup X_n \cup X_{n+1} & \to B_n
\end{array}
$$

are defined such that

$$
\begin{array}{rrcl}
(*) & g_n(h_n(b)) & = & b \textit{ for } b \in B_{n-1} \cup Y_n \\
 & h_n(g_n(a)) & = & a \textit{ for } a \in A_{n-1} \cup X_n \\
(**) & h_n(Y_{n+1}) & = & A_n \setminus (A_{n-1} \cup X_n) \\
 & g_n(X_{n+1}) & = & B_n \setminus (B_{n-1} \cup Y_n).
\end{array}
$$

Note that $(**)$ implies that $h_n(B_{n-1} \cup Y_n) = A_{n-1} \cup X_n$ and $g_n(A_{n-1} \cup X_n) = B_{n-1} \cup Y_n$. We define $\mathcal{A}_{n+1}^{\Sigma} := SH_{\mathcal{M}_\infty}^{\mathcal{A}_\infty}(\bigcup_{i=0}^{n+1} X_i)^{\Sigma}$ and $\mathcal{B}_{n+1}^{\Delta} = SH_{\mathcal{N}_\infty}^{\mathcal{B}_\infty}(\bigcup_{i=0}^{n+1} Y_i)^{\Delta}$ and select subsets $Y_{n+2} \subseteq Y_\infty$ and $X_{n+2} \subseteq X_\infty$ such that $Y_{n+2} \cap \bigcup_{i=0}^{n+1} Y_i = \emptyset = X_{n+2} \cap \bigcup_{i=0}^{n+1} X_i$. In addition, the cardinalities must satisfy $|Y_{n+2}| = |A_{n+1} \setminus (A_n \cup X_{n+1})|$ and $|X_{n+2}| = |B_{n+1} \setminus (B_n \cup Y_{n+1})|$, and the remaining complements $Y_\infty \setminus \bigcup_{i=0}^{n+2} Y_i$ and $X_\infty \setminus \bigcup_{i=0}^{n+2} X_i$ must be countably infinite. Let

$$
\begin{array}{rcl}
\upsilon_{n+1}: Y_{n+2} & \to & A_{n+1} \setminus (A_n \cup X_{n+1}), \\
\xi_{n+1}: X_{n+2} & \to & B_{n+1} \setminus (B_n \cup Y_{n+1})
\end{array}
$$

be arbitrary bijections. We define $h_{n+1} := \upsilon_{n+1} \cup g_n^{-1} \cup h_n$ and $g_{n+1} := \xi_{n+1} \cup h_n^{-1} \cup g_n$. In more detail:

$$
h_{n+1}(b) = \begin{cases} \upsilon_{n+1}(b) & \text{for } b \in Y_{n+2} \\ h_n(b) & \text{for } b \in B_{n-1} \cup Y_n \cup Y_{n+1} \\ g_n^{-1}(b) & \text{for } b \in B_n \setminus (B_{n-1} \cup Y_n) \end{cases}
$$

and

$$
g_{n+1}(a) = \begin{cases} \xi_{n+1}(a) & \text{for } a \in X_{n+2} \\ g_n(a) & \text{for } a \in A_{n-1} \cup X_n \cup X_{n+1} \\ h_n^{-1}(a) & \text{for } a \in A_n \setminus (A_{n-1} \cup X_n). \end{cases}
$$

Without loss of generality we may assume (for notational convenience) that the construction eventually covers all atoms in X_∞ and Y_∞; in other words, we assume that $\bigcup_{i=0}^\infty X_i = X_\infty$ and $\bigcup_{i=0}^\infty Y_i = Y_\infty$, and thus $\bigcup_{i=0}^\infty A_i = A_\infty$

and $\bigcup_{i=0}^{\infty} B_i = B_\infty$. We define the limit mappings

$$\begin{aligned} h_\infty &:= \bigcup_{i=0}^{\infty} h_i : B_\infty \to A_\infty, \\ g_\infty &:= \bigcup_{i=0}^{\infty} g_i : A_\infty \to B_\infty. \end{aligned}$$

It is easy to see that h_∞ and g_∞ are bijections that are inverse to each other: in fact, given $b \in B_\infty$ there is a minimal n such that $b \in B_{n-1}$. By $(*)$ it follows that $g_n(h_n(b)) = b$ and thus $g_\infty(h_\infty(b)) = b$. Accordingly, we obtain $h_\infty(g_\infty(a)) = a$ for all $a \in A_\infty$.

The bijections h_∞ and g_∞ may be used to carry the Δ-structure of $\mathcal{B}_\infty^\Delta$ to $\mathcal{A}_\infty^\Sigma$ and to carry the Σ-structure of $\mathcal{A}_\infty^\Sigma$ to $\mathcal{B}_\infty^\Delta$: let f (f') be an n-ary function symbol of Δ (Σ) and $a_1, \ldots, a_n \in A_\infty$ ($b_1, \ldots, b_n \in B_\infty$). We define

$$\begin{aligned} f_{\mathcal{A}_\infty}(a_1, \ldots, a_n) &:= h_\infty(f_{\mathcal{B}_\infty}(g_\infty(a_1), \ldots, g_\infty(a_n))), \\ f'_{\mathcal{B}_\infty}(b_1, \ldots, b_n) &:= g_\infty(f'_{\mathcal{A}_\infty}(h_\infty(b_1), \ldots, h_\infty(b_n))). \end{aligned}$$

Let p (q) be an n-ary predicate symbol of Δ (Σ) and $a_1, \ldots, a_n \in A_\infty$ ($b_1, \ldots, b_n \in B_\infty$). We define

$$\begin{aligned} p_{\mathcal{A}_\infty}[a_1, \ldots, a_n] &:\Longleftrightarrow p_{\mathcal{B}_\infty}[g_\infty(a_1), \ldots, g_\infty(a_n)], \\ q_{\mathcal{B}_\infty}[b_1, \ldots, b_n] &:\Longleftrightarrow q_{\mathcal{A}_\infty}[h_\infty(b_1), \ldots, h_\infty(b_n)]. \end{aligned}$$

With this definition, the mappings h_∞ and g_∞ are inverse isomorphisms between the $(\Sigma \cup \Delta)$-structures $\mathcal{A}_\infty^{\Sigma\cup\Delta}$ and $\mathcal{B}_\infty^{\Sigma\cup\Delta}$. For this reason, it is irrelevant whether we take $\mathcal{A}_\infty^{\Sigma\cup\Delta}$ or $\mathcal{B}_\infty^{\Sigma\cup\Delta}$ as the combined structure defined by the construction. In the following, we shall use $\mathcal{A}_\infty^{\Sigma\cup\Delta}$ as combined structure, and denote it by $\mathcal{A}^\Sigma \otimes B^\Delta$.

Lemma 6.1 *$\mathcal{A}^\Sigma \otimes B^\Delta$ closes the amalgamation base $(X, \mathcal{A}^\Sigma, \mathcal{B}^\Delta)$, i.e., $\mathcal{A}^\Sigma \otimes B^\Delta$ is an amalgamated product of $\mathcal{A}^\Sigma$ and B^Δ.*

Proof. Obviously, Id_A gives the embedding homomorphism from $\mathcal{A}^\Sigma$ to $\mathcal{A}_\infty^{\Sigma\cup\Delta}$. The restriction of h_∞ to $\mathcal{B}^\Delta$ yields an embedding homomorphism from $\mathcal{B}^\Delta$ to $\mathcal{A}_\infty^{\Sigma\cup\Delta}$. Note that the embedding homomorphisms are even 1–1 in this case. These homomorphisms agree on the shared substructure X since $h_\infty(x) = x$ for all $x \in X$ by construction. Thus, $(\mathcal{A}_\infty^{\Sigma\cup\Delta}, Id_A, h_\infty|_B)$ is an amalgamated product of $\mathcal{A}^\Sigma$ and B^Δ. $\square$

6.2 Free amalgamation of strong SC-structures

In order to obtain a better characterization of what the above construction generates, we restrict our attention to strong SC-structures. First, we must define a class of admissible structures. To this purpose we use the algebraic condition of Proposition 3.2:

Definition 6.2 *For strong SC-structures $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ and $(\mathcal{B}^\Delta, \mathcal{N}, X)$, the class of admissible structures, $\mathrm{Adm}(\mathcal{A}^\Sigma, \mathcal{B}^\Delta)$, consists of all structures $\mathcal{C}^{\Sigma\cup\Delta}$ such that for every mapping $g_{X-C} : X \to C$ there exist unique homomorphisms $g^\Sigma_{A-C} : \mathcal{A}^\Sigma \to \mathcal{C}^\Sigma$ and $g^\Delta_{B-C} : \mathcal{B}^\Delta \to \mathcal{C}^\Delta$ extending g_{X-C}.*

Lemma 6.3 *Let $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ and $(\mathcal{B}^\Delta, \mathcal{N}, X)$ be strong SC-structures. Then $\mathcal{A}^\Sigma \otimes B^\Delta$ is in the chosen class $\mathrm{Adm}(\mathcal{A}^\Sigma, \mathcal{B}^\Delta)$ of admissible structures.*

Proof. Let $g_{X-A_\infty} : X \to A_\infty$ be a mapping. By property (a3), there exists a unique[6] homomorphism $g_{A-A_\infty} : \mathcal{A}^\Sigma \to \mathcal{A}^\Sigma_\infty$ that extends g_{X-A_∞}. By property (b3), the mapping $g_{X-B_\infty} := g_{X-A_\infty} \circ g_\infty : X \to B_\infty$ has a unique extension to a Δ-homomorphism $g_{B-B_\infty} : \mathcal{B}^\Delta \to \mathcal{B}^\Delta_\infty$. Thus, $g_{B-A_\infty} := g_{B-B_\infty} \circ h_\infty : B^\Delta \to \mathcal{A}^\Delta_\infty$ is a Δ-homomorphism. Restricted to X, g_{B-A_∞} is equal to to $g_{X-A_\infty} \circ g_\infty \circ h_\infty = g_{X-A_\infty}$, i.e., it is in fact an extension of g_{X-A_∞}. It remains to be shown that this extension is unique. It is easy to see that for any g'_{B-A_∞} extending g_{X-A_∞}, the composition $g'_{B-B_\infty} := g'_{B-A_\infty} \circ g_\infty$ is a homomorphism extending $g_{X-B_\infty} = g_{X-A_\infty} \circ g_\infty$. By property (b3), this implies $g'_{B-A_\infty} \circ g_\infty = g_{B-B_\infty}$, and thus $g'_{B-A_\infty} = g_{B-B_\infty} \circ h_\infty = g_{B-A_\infty}$. □

The lemma shows that, for strong SC-structures, our construction yields an admissible amalgamated product with respect to $Adm(\mathcal{A}^\Sigma, \mathcal{B}^\Delta)$. Before we can prove that this product is in fact the free amalgamated product, we need one more technical lemma.

Lemma 6.4 *Assume that our construction is applied to strong SC-structures $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ and $(\mathcal{B}^\Delta, \mathcal{N}, X)$. Let $\mathcal{D}^{\Sigma\cup\Delta} \in \mathrm{Adm}(\mathcal{A}^\Sigma, \mathcal{B}^\Delta)$ be an admissible structure.*

1. *For every mapping $f_n : \bigcup_{i=0}^n X_i \to D$ there exists a unique homomorphism $f^\Sigma_{A_n-D} : \mathcal{A}^\Sigma_n \to \mathcal{D}^\Sigma$ that extends f_n.*
2. *Moreover, if $f_{n+1} : \bigcup_{i=0}^{n+1} X_i \to D$ extends f_n, then $f^\Sigma_{A_{n+1}-D}$ extends $f^\Sigma_{A_n-D}$.*

[6] The assumption "$(\mathcal{A}^\Sigma, \mathcal{M}, X)$ strong" is necessary to have uniqueness.

3. For every mapping $g_n : \bigcup_{i=0}^{n} Y_i \to D$ there exists a unique homomorphism $f_{B_n-D}^{\Delta} : \mathcal{B}_n^{\Delta} \to \mathcal{D}^{\Delta}$ that extends g_n.

4. Moreover, if $g_{n+1} : \bigcup_{i=0}^{n+1} Y_i \to D$ extends g_n, then $g_{B_{n+1}-D}^{\Delta}$ extends $g_{B_n-D}^{\Delta}$.

Proof. (1) For $n = 0$, the existence of a unique homomorphisms $f_{A_0-D}^{\Sigma}$ extending the given mappings $f_0 : X = X_0 \to D$ follows from the definition of $Adm(\mathcal{A}^{\Sigma}, \mathcal{B}^{\Delta})$.

For $n > 0$, let $\pi : X \to \widehat{X_n} := \bigcup_{i=0}^{n} X_i$ be an arbitrary bijection. By property (a4), π has a unique extension ϕ_π to an isomorphism from $\mathcal{A}^{\Sigma} = SH_{\mathcal{M}_\infty}^{\mathcal{A}_\infty}(X)$ to $\mathcal{A}_n^{\Sigma} = SH_{\mathcal{M}_\infty}^{\mathcal{A}_\infty}(\widehat{X}_n)$. Because of the definition of $Adm(\mathcal{A}^{\Sigma}, \mathcal{B}^{\Delta})$, the mapping $\pi \circ f_n$ has a unique extension to a homomorphism $f_{A-D}^{\Sigma} : \mathcal{A}^{\Sigma} \to \mathcal{D}^{\Sigma}$. Thus, $f_{A_n-D}^{\Sigma} := \phi_\pi^{-1} \circ f_{A-D}^{\Sigma}$ is a homomorphism from $\mathcal{A}_n^{\Sigma}$ to D^{Σ} that extends f_n^{Σ}.

In order to show uniqueness, assume that $\widehat{f}_{A_n-D}^{\Sigma} : \mathcal{A}_n^{\Sigma} \to \mathcal{D}^{\Sigma}$ is another extension of f_n. It follows that $\phi_\pi \circ \widehat{f}_{A_n-D}^{\Sigma}$ extends $\pi \circ f_n$, and thus $f_{A-D}^{\Sigma} = \phi_\pi \circ \widehat{f}_{A_n-D}^{\Sigma}$. Obviously, this implies $\phi_\pi^{-1} \circ f_{A-D}^{\Sigma} = \widehat{f}_{A_n-D}^{\Sigma}$.

(2) Suppose that $f_{n+1} : \bigcup_{i=0}^{n+1} X_i \to D$ extends $f_n : \bigcup_{i=0}^{n} X_i \to D$. The restriction of $f_{A_{n+1}-D}^{\Sigma}$ to $\mathcal{A}_n^{\Sigma}$ is a homomorphism $\mathcal{A}_n^{\Sigma} \to D^{\Sigma}$ that extends f_n. Since there is a unique homomorphism with this property, namely $f_{A_n-D}^{\Sigma}$, it coincides with this homomorphism.

(3) and (4) follow by symmetry of our construction. □

Theorem 6.5 *If $(\mathcal{A}^{\Sigma}, \mathcal{M}, X)$ and $(\mathcal{B}^{\Delta}, \mathcal{N}, X)$ are strong SC-structures over disjoint signatures, then $\mathcal{A}_\infty^{\Sigma\cup\Delta}$ is the free amalgamated product of $\mathcal{A}^{\Sigma}$ and $\mathcal{B}^{\Delta}$ over X with respect to the class $\mathrm{Adm}(\mathcal{A}^{\Sigma}, \mathcal{B}^{\Delta})$ of admissible structures defined above.*

Proof. We have already shown that $\mathcal{A}_\infty^{\Sigma\cup\Delta}$ is an admissible amalgamated product of $\mathcal{A}^{\Sigma}$ and $\mathcal{B}^{\Delta}$. Recall that Id_A is the embedding homomorphism $h_{A-A_\infty} : \mathcal{A}^{\Sigma} \to \mathcal{A}_\infty^{\Sigma\cup\Delta}$, and h_∞ is the embedding homomorphism $h_{B-A_\infty} : \mathcal{B}^{\Delta} \to \mathcal{A}_\infty^{\Sigma\cup\Delta}$.

In order to show that this admissible amalgamated product is free, assume that $\mathcal{D}^{\Sigma\cup\Delta} \in Adm(\mathcal{A}^{\Sigma}, \mathcal{B}^{\Delta})$ is another admissible amalgamated product, i.e., there are homomorphic embeddings $g_{A-D}^{\Sigma} : \mathcal{A}^{\Sigma} \to \mathcal{D}^{\Sigma}$ and $g_{B-D}^{\Delta} : \mathcal{B}^{\Delta} \to \mathcal{D}^{\Delta}$ such that $h_{X-A} \circ g_{A-D}^{\Sigma} = h_{X-B} \circ g_{B-D}^{\Delta}$. The embeddings h_{X-A} and h_{X-B} of the amalgamation base $(X, \mathcal{A}^{\Sigma}, \mathcal{B}^{\Sigma})$ are the identity on X, which implies that g_{A-D} and g_{B-D} coincide on X. Let g_{X-D} denote the restriction of both

g_{A-D} and g_{B-D} to X. Because $\mathcal{D}^{\Sigma\cup\Delta}$ was assumed to be admissible, we know that

($*$) every extension of g_{X-D} to a homomorphism $\mathcal{A}^{\Sigma} \to \mathcal{D}^{\Sigma}$ coincides with g^{Σ}_{A-D},

($**$) every extension of g_{X-D} to a homomorphism $\mathcal{B}^{\Delta} \to \mathcal{D}^{\Delta}$ coincides with g^{Δ}_{B-D}.

We must show that there exists a unique homomorphism

$$h^{\Sigma\cup\Delta}_{A_\infty-D} : \mathcal{A}^{\Sigma\cup\Delta}_{\infty} \to \mathcal{D}^{\Sigma\cup\Delta}$$

such that

$$\begin{array}{rrcl} (\#) & Id_A \circ h^{\Sigma\cup\Delta}_{A_\infty-D} & = & g^{\Sigma}_{A-D}, \\ (\#\#) & h_\infty\mid_B \circ h^{\Sigma\cup\Delta}_{A_\infty-D} & = & g^{\Delta}_{B-D}. \end{array}$$

This situation is illustrated in the figure. In the first part of the proof we

$$\begin{tikzcd}
 & \mathcal{A}^{\Sigma} \arrow[dr, "\mathrm{Id}"] \arrow[drr, "g_{A-D}"] & & \\
X \arrow[ur, "\mathrm{Id}"] \arrow[dr, "\mathrm{Id}"'] & & (\mathcal{A} \otimes \mathcal{B})^{\Sigma\cup\Delta} \arrow[r, "!"] & \mathcal{D}^{\Sigma\cup\Delta} \\
 & \mathcal{B}^{\Delta} \arrow[ur, "h_\infty"] \arrow[urr, "g_{B-D}"'] & &
\end{tikzcd}$$

show that such a homomorphism $h_{A_\infty-D}$ exists. In the second part, we show uniqueness.

(1) It is sufficient to show that the mapping g_{X-D} can be extended to a homomorphism $h^{\Sigma\cup\Delta}_{A_\infty-D} : \mathcal{A}^{\Sigma\cup\Delta}_{\infty} \to \mathcal{D}^{\Sigma\cup\Delta}$. In fact, it is easy to see that in this case $Id_A \circ h_{A_\infty-D}$ is a homomorphism from $\mathcal{A}^{\Sigma}$ to $\mathcal{D}^{\Sigma}$ that extends g_{X-D}, and $h_\infty\mid_B \circ h_{A_\infty-D}$ is a homomorphism from $\mathcal{B}^{\Delta}$ to $\mathcal{D}^{\Delta}$ that extends g_{X-D}. Thus ($\#$) and ($\#\#$) are immediate consequences of ($*$) and ($**$), respectively.

In order to construct an appropriate homomorphism $h^{\Sigma\cup\Delta}_{A_\infty-D} : \mathcal{A}^{\Sigma\cup\Delta}_{\infty} \to \mathcal{D}^{\Sigma\cup\Delta}$, we define mappings

$$\begin{array}{rcl} h^{\Sigma}_{A_n-D} : A_n & \to & D \\ h^{\Delta}_{B_n-D} : B_n & \to & D \end{array}$$

that satisfy the following properties:

1. $h^{\Sigma}_{A_n-D}$ is a Σ-homomorphism and $h^{\Delta}_{B_n-D}$ is a Δ-homomorphism.

2. If $n > 0$ then, for all $x \in \bigcup_{i=1}^{n} X_i$,

$$h^{\Sigma}_{A_n-D}(x) = h^{\Delta}_{B_{n-1}-D}(g_\infty(x)),$$

and, for all $y \in \bigcup_{i=1}^{n} Y_i$,

$$h^{\Delta}_{B_n-D}(y) = h^{\Sigma}_{A_{n-1}-D}(h_\infty(y)).$$

3. If $n > 0$ then the restriction of $h^{\Sigma}_{A_n-D}$ to A_{n-1} yields $h^{\Sigma}_{A_{n-1}-D}$ and the restriction of $h^{\Delta}_{B_n-D}$ to B_{n-1} yields $h^{\Delta}_{B_{n-1}-D}$.

4. For all $x \in X$, $h^{\Sigma}_{A_n-D}(x) = g_{X-D}(x) = h^{\Delta}_{B_n-D}(x)$.

$n = 0$: Recall that $X_0 = X = Y_0$. By Lemma 6.4, there exist unique extensions of g_{X-D} to homomorphisms

$$\begin{aligned} h^{\Sigma}_{A_0-D} : \mathcal{A}_0 &\rightarrow \mathcal{D}, \\ h^{\Delta}_{B_0-D} : \mathcal{B}_0 &\rightarrow \mathcal{D}. \end{aligned}$$

Obviously, Conditions 1–4 are satisfied.

$n \rightarrow n+1$: Assume that mappings $h^{\Sigma}_{A_n-D}$ and $h^{\Delta}_{B_n-D}$ satisfying Conditions 1–4 are given. We define mappings $f^{\Sigma}_{n+1} : \bigcup_{i=0}^{n+1} X_i \rightarrow D$ and $f^{\Delta}_{n+1} : \bigcup_{i=0}^{n+1} Y_i \rightarrow D$ by

$$\begin{aligned} f^{\Sigma}_{n+1}(x) &= \begin{cases} h^{\Delta}_{B_n-D}(g_\infty(x)) & \text{if } x \in X_{n+1} \\ h^{\Sigma}_{A_n-D}(x) & \text{else,} \end{cases} \\ f^{\Delta}_{n+1}(y) &= \begin{cases} h^{\Sigma}_{A_n-D}(h_\infty(y)) & \text{if } y \in Y_{n+1} \\ h^{\Delta}_{B_n-D}(y) & \text{else.} \end{cases} \end{aligned}$$

By Lemma 6.4, there exists a unique extension of f^{Σ}_{n+1} to a Σ-homomorphism $h^{\Sigma}_{A_{n+1}-D} : \mathcal{A}_{n+1} \rightarrow \mathcal{D}$, and a unique extension of f^{Δ}_{n+1} to a Δ-homomorphism $h^{\Delta}_{B_{n+1}-D} : \mathcal{B}_{n+1} \rightarrow \mathcal{D}$. In addition, these homomorphisms extend $h^{\Sigma}_{A_n-D}$ and $h^{\Delta}_{B_n-D}$, respectively. Thus Conditions 1, 3 and 4 are again satisfied. Without loss of generality, we prove Condition 2 only for $h^{\Sigma}_{A_{n+1}-D}$. For $x \in X_{n+1}$, the condition is satisfied by definition of $f^{\Sigma}_{n+1}(x)$. For $x \in \bigcup_{i=0}^{n} X_i$ we have $h^{\Sigma}_{A_{n+1}-D}(x) = f^{\Sigma}_{n+1}(x) = h^{\Sigma}_{A_n-D}(x)$. By assumption, we know $h^{\Sigma}_{A_n-D}(x) = h^{\Delta}_{B_{n-1}-D}(g_\infty(x))$. Looking back at the definition of $\mathcal{A}^{\Sigma\cup\Delta}_\infty$, we see that $g_\infty(x)$ is an element of B_{n-1}. By assumption, we know that $h^{\Delta}_{B_{n-1}-D}$ and $h^{\Delta}_{B_n-D}$ agree on B_{n-1}.

This completes the construction of the mappings $h^{\Sigma}_{A_n-D}$ and $h^{\Delta}_{B_n-D}$ ($n \geq 0$). Because of Condition 3, we know that $(h^{\Sigma}_{A_n-D})_{n\geq 0}$ and $(h^{\Delta}_{B_n-D})_{n\geq 0}$ are ascending chains of mappings. Thus there exist limit mappings $h^{\Sigma}_{A_\infty - D}$: $A_\infty \to D$ and $h^{\Delta}_{B_\infty - D} : B_\infty \to D$. Obviously, the restriction of $h^{\Sigma}_{A_\infty - D}$ to A_n coincides with $h^{\Sigma}_{A_n - D}$ (resp. the restriction of $h^{\Delta}_{B_\infty - D}$ to B_n coincides with $h^{\Delta}_{B_n - D}$).

It is easy to see that $h_{A_\infty - D}$ is a Σ-homomorphism and $h_{B_\infty - D}$ is a Δ-homomorphism. For instance, assume that f is an n-ary function symbol in Σ, and that $a_1, \ldots, a_n \in A_\infty = \bigcup_{i=0}^{\infty} A_i$. Thus, there exists $k \geq 0$ such that $a_1, \ldots, a_n \in A_k$. By Lemma 4.3, we know that $\mathcal{A}_k = SH^{\Sigma}_{\mathcal{M}_\infty}(\bigcup_{i=0}^{k} X_i)$ is a substructure of $\mathcal{A}_\infty$, and thus $f_{\mathcal{A}_\infty}(a_1, \ldots, a_n) \in A_k$. Since $h^{\Sigma}_{A_\infty - D}$ coincides with $h^{\Sigma}_{A_k - D}$ on A_k, we obtain

$$
\begin{aligned}
h_{A_\infty - D}(f_{\mathcal{A}_\infty}(a_1, \ldots, a_n)) &= h_{A_k - D}(f_{\mathcal{A}_k}(a_1, \ldots, a_n)) \\
&= f_{\mathcal{D}}(h_{A_k - D}(a_1), \ldots, h_{A_k - D}(a_n)) \\
&= f_{\mathcal{D}}(h_{A_\infty - D}(a_1), \ldots, h_{A_\infty - D}(a_n)).
\end{aligned}
$$

It remains to be shown that $h_{A_\infty - D}$ and $h_{B_\infty - D}$ are even $(\Sigma \cup \Delta)$-homomorphisms. In order to show this we prove the following claim:

$$
(\dagger) \quad h_\infty \circ h^{\Sigma}_{A_\infty - D} = h^{\Delta}_{B_\infty - D} \quad \text{and} \quad g_\infty \circ h^{\Delta}_{B_\infty - D} = h^{\Sigma}_{A_\infty - D}.
$$

From the second identity of ($\dagger$) we can easily deduce that $h^{\Sigma}_{A_\infty - D}$ is a $(\Sigma \cup \Delta)$-homomorphism. In fact, we already know that it is a Σ-homomorphism. In addition, $h^{\Delta}_{B_\infty - D}$ is a Δ-homomorphism and g_∞ is a $(\Sigma \cup \Delta)$-homomorphism. Thus the composition $g_\infty \circ h^{\Delta}_{B_\infty - D}$ is a Δ-homomorphism. Accordingly, the first identity of ($\dagger$) implies that $h^{\Delta}_{B_\infty - D}$ is a $(\Sigma \cup \Delta)$-homomorphism.

To complete Part 1 of the proof, we show the first identity of ($\dagger$). (The second follows by symmetry.) Let b be an element of B_∞. Thus there is an $n \geq 0$ such that $b \in B_n \setminus B_{n-1}$. First, assume that $b \in Y_n$. By construction of $\mathcal{A}^{\Sigma \cup \Delta}_\infty$, this implies $h_\infty(b) \in A_{n-1}$, and thus we have

$$
h^{\Sigma}_{A_\infty - D}(h_\infty(b)) = h^{\Sigma}_{A_{n-1} - D}(h_\infty(b)) = h^{\Delta}_{B_n - D}(b) = h^{\Delta}_{B_\infty - D}(b).
$$

The second identity holds by Condition 2 in the construction of the mappings $h^{\Delta}_{B_n - D}$ and $h^{\Sigma}_{A_n - D}$, and the third follows from the definition of $h^{\Delta}_{B_\infty - D}$.

Second, assume that $b \in B_n \setminus (B_{n-1} \cup Y_n)$. In this case we have $h_\infty(b) = g^{-1}_\infty(b) \in X_{n+1}$, and thus

$$
\begin{aligned}
h^{\Sigma}_{A_\infty - D}(h_\infty(b)) &= h^{\Sigma}_{A_{n+1} - D}(g^{-1}_\infty(b)) \\
&= h^{\Delta}_{B_n - D}(g_\infty(g^{-1}_\infty(b))) = h^{\Delta}_{B_n - D}(b) \\
&= h^{\Delta}_{B_\infty - D}(b).
\end{aligned}
$$

To sum up, we have shown the existence of a $(\Sigma \cup \Delta)$-homomorphism $h^{\Sigma}_{A_\infty - D}$ that extends g_{X-D}, which completes the first part of the proof.

(2) In order to show uniqueness, assume that there exists a $(\Sigma \cup \Delta)$-homomorphism $h'_{A_\infty - D}$ such that

$$h'^{\Sigma\cup\Delta}_{A_\infty - D} : \mathcal{A}^{\Sigma\cup\Delta}_{\infty} \to \mathcal{D}^{\Sigma\cup\Delta}$$

such that

$$\begin{array}{rrcl} (\#') & Id_A \circ h'^{\Sigma\cup\Delta}_{A_\infty - D} & = & g^{\Sigma}_{A-D}, \\ (\#'\#') & h_\infty \mid_B \circ h'^{\Sigma\cup\Delta}_{A_\infty - D} & = & g^{\Delta}_{B-D}. \end{array}$$

Let $h'^{\Sigma\cup\Delta}_{B_\infty - D} := h_\infty \circ h'^{\Sigma\cup\Delta}_{A_\infty - D}$. It follows that $h'^{\Sigma\cup\Delta}_{A_\infty - D} = g_\infty \circ h'^{\Sigma\cup\Delta}_{B_\infty - D}$. By induction on n we shall show that $h'^{\Sigma\cup\Delta}_{A_\infty - D}$ and $h^{\Sigma\cup\Delta}_{A_\infty - D}$ coincide on A_n, and that $h'^{\Sigma\cup\Delta}_{B_\infty - D}$ and $h^{\Sigma\cup\Delta}_{B_\infty - D}$ coincide on B_n. This implies that $h'^{\Sigma\cup\Delta}_{A_\infty - D}$ and $h^{\Sigma\cup\Delta}_{A_\infty - D}$ coincide on $A_\infty = \bigcup_{n=0}^{\infty} A_n$.

$n = 0$. The conditions $(\#')$ and $(\#'\#')$ imply that the restriction of $h'^{\Sigma\cup\Delta}_{A_\infty - D}$ to $A = A_0$ coincides with g_{A-D}, and the restriction of $h'^{\Sigma\cup\Delta}_{B_\infty - D}$ to $B = B_0$ coincides with g_{B-D}. Thus, both coincide with g_{X-D} on X. Since, by Lemma 6.4, there exist unique extensions of g_{X-D} to homomorphisms $\mathcal{A}_0 \to \mathcal{D}$ and $\mathcal{B}_0 \to \mathcal{D}$, we are done.

$n \to n+1$. Suppose that $h'^{\Sigma\cup\Delta}_{A_\infty - D}$ and $h^{\Sigma\cup\Delta}_{A_\infty - D}$ coincide on A_n, and that $h'^{\Sigma\cup\Delta}_{B_\infty - D}$ and $h^{\Sigma\cup\Delta}_{B_\infty - D}$ coincide on B_n. For $x \in X_{n+1}$ we have $g_\infty(x) \in B_n$, and thus $h'^{\Sigma\cup\Delta}_{A_\infty - D}(x) = h'^{\Sigma\cup\Delta}_{B_\infty - D}(g_\infty(x)) = h^{\Sigma\cup\Delta}_{B_\infty - D}(g_\infty(x)) = h^{\Sigma\cup\Delta}_{A_\infty - D}(x)$. Thus $h'^{\Sigma\cup\Delta}_{A_\infty - D}$ and $h^{\Sigma\cup\Delta}_{A_\infty - D}$ also coincide on $\bigcup_{i=0}^{n+1} X_i$. It follows from Lemma 6.4 that both homomorphisms coincide on A_{n+1}. Similarly, it can be shown that $h'^{\Sigma\cup\Delta}_{B_\infty - D}$ and $h^{\Sigma\cup\Delta}_{B_\infty - D}$ coincide on B_{n+1}. $\square$

For strong SC-structures, the amalgamation construction can be applied iteratedly because the obtained structure is again a strong SC-structure:

Theorem 6.6 *The free amalgamated product of two strong SC-structures with common atom set X is a strong SC-structure with atom set X.*

Proof. We must show that $(\mathcal{A}^{\Sigma\cup\Delta}_{\infty}, End^{\Sigma\cup\Delta}_{\mathcal{A}_\infty}, X)$ is an SC-structure. If we choose $\mathcal{D}^{\Sigma\cup\Delta} = \mathcal{A}^{\Sigma\cup\Delta}_{\infty}$, the first part of the previous proof shows that every mapping $h_{X-A_\infty} : X \to A_\infty$ can be extended to an endomorphism of $\mathcal{A}^{\Sigma\cup\Delta}_{\infty}$. Thus X is an atom set for $\mathcal{A}^{\Sigma\cup\Delta}_{\infty}$. It remains to be shown that every element $a \in A_\infty$ is stabilized—with respect to $End^{\Sigma\cup\Delta}_{\mathcal{A}_\infty}$—by a finite subset of X. By induction on n ($n \geq 0$) we shall show that every $a \in A_n$ and every $b \in B_n$ is stabilized—with respect to $End^{\Sigma\cup\Delta}_{\mathcal{A}_\infty}$ and $End^{\Sigma\cup\Delta}_{\mathcal{B}_\infty}$, respectively—by a finite subset of X.

$n = 0$. Let $a \in A_0 = SH^{\mathcal{A}_\infty}_{\mathcal{M}_\infty}(X)$. Thus a is stabilized by $X = X_0$ with respect to $End^{\Sigma}_{\mathcal{A}_\infty}$. In addition, since $(\mathcal{A}^{\Sigma}_\infty, End^{\Sigma}_{\mathcal{A}_\infty}, X_\infty)$ is a strong SC-structure, a is stabilized by a finite subset of X_∞. Both facts together imply that the stabilizer of a with respect to $End^{\Sigma}_{\mathcal{A}_\infty}$ is a finite subset, say Z, of $X = X_0$. Since every $(\Sigma \cup \Delta)$-endomorphism is a Σ-endomorphism, Z also stabilizes a with respect to $End^{\Sigma\cup\Delta}_{\mathcal{A}_\infty}$. A symmetric argument shows that every $b \in B_0 = SH^{\mathcal{B}_\infty}_{\mathcal{N}_\infty}(X)$ is stabilized by a finite subset of $X = Y_0$ with respect to $End^{\Sigma\cup\Delta}_{\mathcal{B}_\infty}$.

$n \to n+1$. Suppose that every $a' \in A_n$ and every $b' \in B_n$ is stabilized—with respect to $End^{\Sigma\cup\Delta}_{\mathcal{A}_\infty}$ and $End^{\Sigma\cup\Delta}_{\mathcal{B}_\infty}$ respectively—by a finite subset of X. For $a \in A_{n+1}$, let Z denote the stabilizer of a with respect to $End^{\Sigma}_{\mathcal{A}_\infty}$. Thus, Z is finite, and as in the case "$n = 0$" one can deduce $Z \subseteq \bigcup_{i=0}^{n+1} X_i$. It is easy to see that $Z' := g_\infty(Z)$ stabilizes $b := g_\infty(a)$ with respect to $End^{\Sigma}_{\mathcal{B}_\infty}$, and thus also with respect to $End^{\Sigma\cup\Delta}_{\mathcal{B}_\infty}$. By definition of the mapping g_∞, we know that $Z' \subseteq B_n$, and thus we can apply the induction hypothesis. This yields a finite set $R \subseteq X$ that stabilizes all elements of Z' with respect to $End^{\Sigma\cup\Delta}_{\mathcal{B}_\infty}$. Consequently, R stabilizes b with respect to $End^{\Sigma\cup\Delta}_{\mathcal{B}_\infty}$. It follows that $h_\infty(R) = R \subseteq X$ stabilizes $a = h_\infty(b)$ with respect to $End^{\Sigma\cup\Delta}_{\mathcal{A}_\infty}$. Thus, we have shown that every element of A_{n+1} is stabilized by a finite subset of X with respect to $End^{\Sigma\cup\Delta}_{\mathcal{A}_\infty}$. Symmetrically, one can prove that every element of B_{n+1} is stabilized by a finite subset of X with respect to $End^{\Sigma\cup\Delta}_{\mathcal{B}_\infty}$. □

Obviously, the set of admissible structures, as introduced in Definition 6.2 above, satisfies $Adm(\mathcal{A}^{\Sigma}, \mathcal{B}^{\Delta}) = Adm(\mathcal{B}^{\Delta}, \mathcal{A}^{\Sigma})$. Thus, the amalgamation construction is commutative. In order to show associativity, we must prove that the assumptions of Theorem 3.17 are satisfied.

First, we extend the definition of the class of admissible structures to the case of the simultaneous amalgamation of three structures: For strong SC-structures $(\mathcal{B}^{\Sigma_i}_i, \mathcal{M}_i, X)$ $(i = 1, 2, 3)$, the *class of admissible structures*, $Adm(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$, consists of all structures $\mathcal{C}^{\Sigma_1\cup\Sigma_2\cup\Sigma_3}$ such that for every mapping $g_{X-C} : X \to C$ there exist unique homomorphisms $g^{\Sigma_i}_{B_i-C} : \mathcal{B}^{\Sigma_i}_i \to \mathcal{C}^{\Sigma_i}$ $(i = 1, 2, 3)$ extending g_{X-C}. As an obvious consequence of this definition we obtain:

Lemma 6.7 $Adm(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3) \subseteq Adm(\mathcal{B}_1, \mathcal{B}_2) \cap Adm(\mathcal{B}_2, \mathcal{B}_3)$.

Thus, we have proved that the assumptions of Theorem 3.17 are satisfied, as soon as we have shown the next two lemmas.

Lemma 6.8 $Adm(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3) \subseteq Adm(\mathcal{B}_1, \mathcal{B}_2 \otimes \mathcal{B}_3) \cap Adm(\mathcal{B}_1 \otimes \mathcal{B}_2, \mathcal{B}_3)$.

Proof. We show $Adm(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3) \subseteq Adm(\mathcal{B}_1, \mathcal{B}_2 \otimes \mathcal{B}_3)$. (The other inclusion follows by symmetry.) Thus, assume that $\mathcal{C} \in Adm(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$, and that $g : X \to C$ is given. By definition of $Adm(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$, the mapping g can uniquely be extended to homomorphisms $g_{B_i - C} : \mathcal{B}_i \to \mathcal{C}$ (for $i = 1, 2, 3$).

Now, we apply the amalgamation construction to $\mathcal{B}_2$ and $\mathcal{B}_3$, which yields the free amalgamated product $\mathcal{B}_{23} := \mathcal{B}_2 \otimes \mathcal{B}_3$. Since the common part X of $\mathcal{B}_2$ and $\mathcal{B}_3$ is embedded via Id_X, the embedding homomorphisms $h_{B_i - B_{23}} : \mathcal{B}_i \to \mathcal{B}_{23}$ of this product satisfy $h_{B_2 - B_{23}}|_X = h_{B_3 - B_{23}}|_X$, i.e., their restriction to X coincide. By construction, this restriction to X coincides with Id_X, which means that we have

$$h_{B_2 - B_{23}}|_X = Id_X = h_{B_3 - B_{23}}|_X. \tag{6.9}$$

By Lemma 6.7, $\mathcal{C}$ is also an element of $Adm(\mathcal{B}_2, \mathcal{B}_3)$. In addition, the embedding homomorphisms $g_{B_2 - C} : \mathcal{B}_2 \to \mathcal{C}$ and $g_{B_3 - C} : \mathcal{B}_3 \to \mathcal{C}$ satisfy $Id_X \circ g_{B_2 - C} = g_{B_2 - C}|_X = g = g_{B_3 - C}|_X = Id_X \circ g_{B_3 - C}$, which shows that $\mathcal{C}$ is an admissible amalgamated product of $\mathcal{B}_2$ and $\mathcal{B}_3$. Since $\mathcal{B}_{23}$ is the free amalgamated product, there exists a unique homomorphism $h_{B_{23} - C} : \mathcal{B}_{23} \to \mathcal{C}$ such that

$$g_{B_i - C} \quad = \quad h_{B_i - B_{23}} \circ h_{B_{23} - C} \quad (i = 2, 3). \tag{6.10}$$

We show that the restriction of $h_{B_{23} - C}$ to X coincides with g. In fact,

$$h_{B_{23} - C}|_X = (h_{B_2 - B_{23}} \circ h_{B_{23} - C})|_X = g_{B_2 - C}|_X = g.$$

The first identity holds because of (6.9), the second because of (6.10), and the third because $g_{B_2 - C}$ extends g. This shows that there exists an extension of g to a homomorphism from $\mathcal{B}_{23}$ to $\mathcal{C}$.

In order to prove $\mathcal{C} \in Adm(\mathcal{B}_1, \mathcal{B}_2 \otimes \mathcal{B}_3)$, it remains to be shown that this extension is unique. Thus, assume that $f_{B_{23} - C} : \mathcal{B}_{23} \to \mathcal{C}$ is another homomorphism that extends g. Because of (6.9), we can deduce that the composition $h_{B_2 - B_{23}} \circ f_{B_{23} - C}$ is a homomorphism of $\mathcal{B}_2$ into $\mathcal{C}$ that extends g. Since $g_{B_2 - C}$ is unique with this property, we obtain

$$g_{B_2 - C} \quad = \quad h_{B_2 - B_{23}} \circ f_{B_{23} - C}. \tag{6.11}$$

Similarly, it can be shown that

$$g_{B_3 - C} \quad = \quad h_{B_3 - B_{23}} \circ f_{B_{23} - C}. \tag{6.12}$$

Because $h_{B_{23} - C}$ is the unique homomorphism satisfying (6.10), the identities (6.11) and (6.12) imply $f_{B_{23} - C} = h_{B_{23} - C}$. □

Lemma 6.13 $\{\mathcal{B}_1 \otimes (\mathcal{B}_2 \otimes \mathcal{B}_3), (\mathcal{B}_1 \otimes \mathcal{B}_2) \otimes \mathcal{B}_3\} \subseteq Adm(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$.

Proof. We show $\mathcal{B}_1 \otimes (\mathcal{B}_2 \otimes \mathcal{B}_3) \in Adm(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$. (The other inclusion follows by symmetry.) As before, we denote $\mathcal{B}_1 \otimes (\mathcal{B}_2 \otimes \mathcal{B}_3)$ by $\mathcal{B}_{123}$ and $\mathcal{B}_2 \otimes \mathcal{B}_3$ by $\mathcal{B}_{23}$.

Let $g : X \to B_{123}$ be a mapping. We know that $\mathcal{B}_{123} = \mathcal{B}_1 \otimes (\mathcal{B}_2 \otimes \mathcal{B}_3)$ is an element of $Adm(\mathcal{B}_1, \mathcal{B}_2 \otimes \mathcal{B}_3)$, and thus there exists a unique Σ_1-homomorphism $g_{B_1-B_{123}} : \mathcal{B}_1 \to \mathcal{B}_{123}$ that extends g.

As a $(\Sigma_2 \cup \Sigma_3)$-structure, $\mathcal{B}_{23}$ is isomorphic to $\mathcal{B}_{123}$ (by property (b0) in the construction). Let $h^{\Sigma_2 \cup \Sigma_3}_{B_{23}-B_{123}}$ be the corresponding isomorphism, and let $k^{\Sigma_2 \cup \Sigma_3}_{B_{123}-B_{23}}$ be its inverse. We consider the mapping $g' = g \circ k_{B_{123}-B_{23}} : X \to B_{23}$. Since $\mathcal{B}_{23} = \mathcal{B}_2 \otimes \mathcal{B}_3$ is in $Adm(\mathcal{B}_2, \mathcal{B}_3)$, there exist unique extensions of g' to Σ_i-homomorphisms $g_{B_i-B_{23}} : \mathcal{B}_i \to \mathcal{B}_{23}$ (for $i = 2, 3$). Obviously, $g_{B_i-B_{23}} \circ h_{B_{23}-B_{123}}$ is a Σ_i-homomorphism from $\mathcal{B}_i$ to $\mathcal{B}_{123}$ that extends g (for $i = 2, 3$).

It remains to be shown that these extensions are unique. Assume that $f_{B_i-B_{123}} : \mathcal{B}_i \to \mathcal{B}_{123}$ are Σ_i-homomorphisms extending g ($i = 2, 3$). Then $f_{B_i-B_{123}} \circ k_{B_{123}-B_{23}} : \mathcal{B}_i \to \mathcal{B}_{23}$ is a Σ_i-homomorphism extending $g' = g \circ k_{B_{123}-B_{23}}$, and thus uniqueness of $g_{B_i-B_{23}}$ with this property implies $f_{B_i-B_{123}} \circ k_{B_{123}-B_{23}} = g_{B_i-B_{23}}$. It follows that

$$g_{B_i-B_{23}} \circ h_{B_{23}-B_{123}} = f_{B_i-B_{123}} \circ k_{B_{123}-B_{23}} \circ h_{B_{23}-B_{123}} = f_{B_i-B_{123}},$$

which yields the desired uniqueness result. □

To sum up, we have shown that Theorem 3.17 can be applied, which yields:

Theorem 6.14 *Free amalgamation of strong SC-structures with disjoint signatures over the same atom set is associative.*

7 Combining Constraint Solvers for arbitrary SC-Structures: The Existential Positive Case

Let $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ and $(\mathcal{B}^\Delta, \mathcal{N}, X)$ be two SC-structures over disjoint signatures Σ and Δ; let $\mathcal{A}^\Sigma \otimes B^\Delta = \mathcal{A}^{\Sigma \cup \Delta}_\infty$ denote their amalgamated product, as constructed in the previous section. In this section we shall prove the following result.

Theorem 7.1 *The existential positive theory of $\mathcal{A}^\Sigma \otimes \mathcal{B}^\Delta$ is decidable, provided that the positive theories of $\mathcal{A}^\Sigma$ and of $\mathcal{B}^\Delta$ are decidable.*

Note that this theorem holds for arbitrary SC-structures, i.e., it is not required that $\mathcal{A}^\Sigma$ and $\mathcal{B}^\Delta$ are strong. In this general setting, however, it is not yet clear in which sense the amalgamated product $\mathcal{A}^\Sigma \otimes \mathcal{B}^\Delta$ obtained by our construction plays a unique rôle among all possible closures of the amalgamation base $(X, \mathcal{A}^\Sigma, \mathcal{B}^\Delta)$. For strong SC-structures we know that $\mathcal{A}^\Sigma \otimes \mathcal{B}^\Delta$ is the free amalgamated product.

7.1 The decomposition algorithm

The decomposition algorithm described below decomposes a positive existential $(\Sigma \cup \Delta)$-sentence φ_0 into a finite set of pairs (α, β), where α is a positive Σ-sentence and β is a positive Δ-sentence. This algorithm coincides with the one described in [BaS94a], where it has been used in the restricted context of combination problems for free structures.

Before we can describe the algorithm, we must introduce some notation. In the following, V denotes an infinite set of variables used by the first order languages under consideration. Let t be a $(\Sigma \cup \Delta)$-term. This term is called *pure* iff it is either a Σ-term or a Δ-term. An equation is pure iff it is an equation between pure terms of the same signature. A relational formula $p[s_1, \ldots, s_m]$ is pure iff $s_1, \ldots, s_m$ are pure terms of the signature of p. Now assume that t is a non-pure term whose topmost function symbol is in Σ. A subterm s of t is called *alien subterm* of t iff its topmost function symbol belongs to Δ and every proper superterm of s in t has its top symbol in Σ. Alien subterms of terms with top symbol in Δ are defined analogously. For a relational formula $p[s_1, \ldots, s_m]$, alien subterms are defined as follows: if s_i has a top symbol whose signature is different from the signature of p then s_i itself is an alien subterm; otherwise, any alien subterm of s_i is an alien subterm of $p[s_1, \ldots, s_m]$.

Algorithm 1

Let φ_0 be a positive existential $(\Sigma \cup \Delta)$-sentence. Without loss of generality, we may assume that φ_0 has the form $\exists \vec{u}_0\ \gamma_0$, where γ_0 is a conjunction of atomic formulae. Indeed, since existential quantifiers distribute over disjunction, a sentence $\exists \vec{u}_0\ (\gamma_1 \vee \gamma_2)$ is valid iff $\exists \vec{u}_0\ \gamma_1$ or $\exists \vec{u}_0\ \gamma_2$ is valid.

Step 1: Transform non-pure atomic formulae.

(1) Equations $s = t$ of γ_0 where s and t have topmost function symbols belonging to different signatures are replaced by (the conjunction of) two new equations $u = s, u = t$, where u is a new variable. The quantifier prefix is extended by adding an existential quantification for u.

(2) As a result, we may assign a unique label Σ or Δ to each atomic formula that is not an equation between variables. The label of an equation $s = t$ is the signature of the topmost function symbols of s and/or t. The label of a relational formula $p[s_1, \ldots, s_m]$ is the signature of p.

(3) Now alien subterms occurring in atomic formulae are successively replaced by new variables. For example, assume that $s = t$ is an equation in the current formula, and that s contains the alien subterm s_1. Let u be a variable not occurring in the current formula, and let s' be the term obtained from s by replacing s_1 by u. Then the original equation is replaced by (the conjunction of) the two equations $s' = t$ and $u = s_1$. The quantifier prefix is extended by adding an existential quantification for u. The equation $s' = t$ keeps the label of $s = t$, and the label of $u = s_1$ is the signature of the top symbol of s_1. Relational atomic formulae with alien subterms are treated analogously. This process is iterated until all atomic formulae occurring in the conjunctive matrix are pure. It is easy to see that this is achieved after finitely many iterations.

Step 2: Remove atomic formulae without label.

Equations between variables occurring in the conjunctive matrix are removed as follows: If $u = v$ is such an equation then one removes $\exists u$ from the quantifier prefix and $u = v$ from the matrix. In addition, every occurrence of u in the remaining matrix is replaced by v. This step is iterated until the matrix contains no equations between variables.

Let φ_1 be the new sentence obtained this way. The matrix of φ_1 can be written as a conjunction $\gamma_{1,\Sigma} \wedge \gamma_{1,\Delta}$, where $\gamma_{1,\Sigma}$ is a conjunction of all atomic formulae from φ_1 with label Σ, and $\gamma_{1,\Delta}$ is a conjunction of all atomic formulae from φ_1 with label Δ. There are three different types of variables occurring in φ_1: shared variables occur both in $\gamma_{1,\Sigma}$ and in $\gamma_{1,\Delta}$; Σ-variables occur only in $\gamma_{1,\Sigma}$; and Δ-variables occur only in $\gamma_{1,\Delta}$. Let $\vec{u}_{1,\Sigma}$ be the tuple of all Σ-variables, $\vec{u}_{1,\Delta}$ be the tuple of all Δ-variables, and $\vec{u}_1$ be the tuple of all shared variables.[7] Obviously, φ_1 is equivalent to the sentence

$$\exists\vec{u}_1\left(\exists\vec{u}_{1,\Sigma}\ \gamma_{1,\Sigma} \wedge \exists\vec{u}_{1,\Delta}\ \gamma_{1,\Delta}\right).$$

The next two steps of the algorithm are nondeterministic, i.e., a given sentence is transformed into finitely many new sentences. Here the idea is that the original sentence is valid iff at least one of the new sentences is valid.

Step 3: Variable identification.

Consider all possible partitions of the set of all shared variables. Each of these partitions yields one of the new sentences as follows. The variables in each class of the partition are "identified" with each other by choosing an element of the class as representative, and replacing in the sentence all occurrences of variables of the class by this representative. Quantifiers for replaced variables are removed.

Let $\exists\vec{u}_2\,(\exists\vec{u}_{1,\Sigma}\ \gamma_{2,\Sigma} \wedge \exists\vec{u}_{1,\Delta}\ \gamma_{2,\Delta})$ denote one of the sentences obtained by Step 3.

Step 4: Choose signature labels and ordering.

We choose a label Σ or Δ for every (shared) variable in $\vec{u}_2$, and a linear ordering $<$ on these variables.

For each of the choices made in Step 3 and 4, the algorithm yields a pair (α, β) of sentences as output.

Step 5: Generate output sentences.

The sentence $\exists\vec{u}_2(\exists\vec{u}_{1,\Sigma}\ \gamma_{2,\Sigma} \wedge \exists\vec{u}_{1,\Delta}\ \gamma_{2,\Delta})$ is split into two sentences

$$\alpha = \forall\vec{v}_1\exists\vec{w}_1\ldots\forall\vec{v}_k\exists\vec{w}_k\exists\vec{u}_{1,\Sigma}\ \gamma_{2,\Sigma}$$

and

$$\beta = \exists\vec{v}_1\forall\vec{w}_1\ldots\exists\vec{v}_k\forall\vec{w}_k\exists\vec{u}_{1,\Delta}\ \gamma_{2,\Delta}.$$

Here $\vec{v}_1\vec{w}_1\ldots\vec{v}_k\vec{w}_k$ is the unique re-ordering of $\vec{u}_2$ along $<$. The variables $\vec{v}_i$ ($\vec{w}_i$) are the variables with label Δ (label Σ).

Thus, the overall output of the algorithm is a finite set of pairs of sentences. Note that the sentences α and β are positive formulae, but they need no longer be existential positive formulae.

Obviously, Theorem 7.1 follows immediately as soon as we have shown that the decomposition algorithm is sound and complete.

[7] The order in these tuples can be chosen arbitrarily.

7.2 Correctness of the Decomposition Algorithm

This proof is very similar to the one given in [BaS94a] for the combination of constraint solvers in free structures. First, we show soundness of the algorithm, i.e., if one of the output pairs is valid then the original sentence was valid.

Lemma 7.2 *$\mathcal{A}_\infty^{\Sigma\cup\Delta} \models \varphi_0$ if $\mathcal{A}^\Sigma \models \alpha$ and $\mathcal{B}^\Delta \models \beta$ for some output pair (α, β).*

Proof. Since $\mathcal{A}^\Sigma$ and $\mathcal{A}_\infty^\Sigma$ are isomorphic Σ-structures (see the points (a0) and (b0) in the amalgamation construction), we know that $\mathcal{A}_\infty^\Sigma \models \alpha$. Accordingly, we also have $\mathcal{B}_\infty^\Delta \models \beta$. Moreover, since $\mathcal{A}_\infty^{\Sigma\cup\Delta}$ and $\mathcal{B}_\infty^{\Sigma\cup\Delta}$ are isomorphic, we know that $\mathcal{A}_\infty^\Delta \models \beta$, i.e., the Δ-reduct of the $(\Sigma\cup\Delta)$-structure $\mathcal{A}_\infty^{\Sigma\cup\Delta}$ satisfies β. This means

$$\begin{array}{rlcl} (*) & \mathcal{A}_\infty^\Sigma & \models & \forall\vec{v}_1\exists\vec{w}_1\ldots\forall\vec{v}_k\exists\vec{w}_k\exists\vec{u}_{1,\Sigma}\ \gamma_{2,\Sigma}(\vec{v}_1,\vec{w}_1,\ldots,\vec{v}_k,\vec{w}_k,\vec{u}_{1,\Sigma}), \\ (**) & \mathcal{A}_\infty^\Delta & \models & \exists\vec{v}_1\forall\vec{w}_1\ldots\exists\vec{v}_k\forall\vec{w}_k\exists\vec{u}_{1,\Delta}\ \gamma_{2,\Delta}(\vec{v}_1,\vec{w}_1,\ldots,\vec{v}_k,\vec{w}_k,\vec{u}_{1,\Delta}). \end{array}$$

Because of the existential quantification over $\vec{v}_1$ in $(**)$, there exist elements $\vec{a}_1 \in \vec{A}_\infty$ such that

$$(***)\quad \mathcal{A}_\infty^\Delta \models \forall\vec{w}_1\ldots\exists\vec{v}_k\forall\vec{w}_k\exists\vec{u}_{1,\Delta}\ \gamma_{2,\Delta}(\vec{a}_1,\vec{w}_1,\ldots,\vec{v}_k,\vec{w}_k,\vec{u}_{1,\Delta}).$$

Because of the universal quantification over $\vec{v}_1$ in $(*)$ we have

$$\mathcal{A}_\infty^\Sigma \models \exists\vec{w}_1\ldots\forall\vec{v}_k\exists\vec{w}_k\exists\vec{u}_{1,\Sigma}\ \gamma_{2,\Sigma}(\vec{a}_1,\vec{w}_1,\ldots,\vec{v}_k,\vec{w}_k,\vec{u}_{1,\Sigma}).$$

Because of the existential quantification over $\vec{w}_1$ in this formula there exist elements $\vec{c}_1 \in \vec{A}_\infty$ such that

$$\mathcal{A}_\infty^\Sigma \models \forall\vec{v}_2\exists\vec{w}_2\ldots\forall\vec{v}_k\exists\vec{w}_k\exists\vec{u}_{1,\Sigma}\ \gamma_{2,\Sigma}(\vec{a}_1,\vec{c}_1,\vec{v}_2,\vec{w}_2,\ldots,\vec{v}_k,\vec{w}_k,\vec{u}_{1,\Sigma}).$$

Because of the universal quantification over $\vec{w}_1$ in $(***)$ we have

$$\mathcal{A}_\infty^\Delta \models \exists\vec{v}_2\forall\vec{w}_2\ldots\exists\vec{v}_k\forall\vec{w}_k\exists\vec{u}_{1,\Delta}\ \gamma_{2,\Delta}(\vec{a}_1,\vec{c}_1,\vec{v}_2,\vec{w}_2,\ldots,\vec{v}_k,\vec{w}_k,\vec{u}_{1,\Delta}).$$

Iterating this argument, we thus obtain

$$\begin{array}{lcl} \mathcal{A}_\infty^\Sigma & \models & \exists\vec{u}_{1,\Sigma}\ \gamma_{2,\Sigma}(\vec{a}_1,\vec{c}_1,\ldots,\vec{a}_k,\vec{c}_k,\vec{u}_{1,\Sigma}), \\ \mathcal{A}_\infty^\Delta & \models & \exists\vec{u}_{1,\Delta}\ \gamma_{2,\Delta}(\vec{a}_1,\vec{c}_1,\ldots,\vec{a}_k,\vec{c}_k,\vec{u}_{1,\Delta}). \end{array}$$

It follows that

$$\mathcal{A}_\infty^{\Sigma\cup\Delta} \models \exists\vec{u}_{1,\Sigma}\ \gamma_{2,\Sigma}(\vec{a}_1,\vec{c}_1,\ldots,\vec{a}_k,\vec{c}_k,\vec{u}_{1,\Sigma}) \wedge \exists\vec{u}_{1,\Delta}\ \gamma_{2,\Delta}(\vec{a}_1,\vec{c}_1,\ldots,\vec{a}_k,\vec{c}_k,\vec{u}_{1,\Delta}).$$

Obviously, this implies that

$$\mathcal{A}_\infty^{\Sigma\cup\Delta} \models \exists\vec{u}_2\,(\exists\vec{u}_{1,\Sigma}\ \gamma_{2,\Sigma} \wedge \exists\vec{u}_{1,\Delta}\ \gamma_{2,\Delta})\,,$$

i.e., one of the sentences obtained after Step 3 of the algorithm holds in $\mathcal{A}_\infty^{\Sigma\cup\Delta}$. It is easy to see that this implies that $\mathcal{A}_\infty^{\Sigma\cup\Delta} \models \varphi_0$. ☐

Next, we show completeness of the decomposition algorithm, i.e., if the input sentence was valid then there exists a valid output pair.

Lemma 7.3 *If $\mathcal{A}_\infty^{\Sigma\cup\Delta} \models \varphi_0$ then $\mathcal{A}^\Sigma \models \alpha$ and $\mathcal{B}^\Delta \models \beta$ for some output pair (α, β).*

Proof. Assume that $\mathcal{A}_\infty^{\Sigma\cup\Delta} \simeq \mathcal{B}_\infty^{\Sigma\cup\Delta} \models \exists\vec{u}_0\gamma_0$. Obviously, this implies that $\mathcal{B}_\infty^{\Sigma\cup\Delta} \models \exists\vec{u}_1\,(\exists\vec{u}_{1,\Sigma}\ \gamma_{1,\Sigma}(\vec{u}_1, \vec{u}_{1,\Sigma}) \wedge \exists\vec{u}_{1,\Delta}\ \gamma_{1,\Delta}(\vec{u}_1, \vec{u}_{1,\Delta}))$, i.e., $\mathcal{B}_\infty^{\Sigma\cup\Delta}$ satisfies the sentence that is obtained after Step 2 of the decomposition algorithm. Thus there exists an assignment $\nu : V \to B_\infty$ such that $\mathcal{B}_\infty^{\Sigma\cup\Delta} \models \exists\vec{u}_{1,\Sigma}\ \gamma_{1,\Sigma}(\nu(\vec{u}_1), \vec{u}_{1,\Sigma}) \wedge \exists\vec{u}_{1,\Delta}\ \gamma_{1,\Delta}(\nu(\vec{u}_1), \vec{u}_{1,\Delta})$.

In Step 3 of the decomposition algorithm, we identify two shared variables u and u' of $\vec{u}_1$ if, and only if, $\nu(u) = \nu(u')$. With this choice, $\mathcal{B}_\infty^{\Sigma\cup\Delta} \models \exists\vec{u}_{1,\Sigma}\ \gamma_{2,\Sigma}(\nu(\vec{u}_2), \vec{u}_{1,\Sigma}) \wedge \exists\vec{u}_{1,\Delta}\ \gamma_{2,\Delta}(\nu(\vec{u}_2), \vec{u}_{1,\Delta})$, and all components of $\nu(\vec{u}_2)$ are distinct.

In Step 4, a shared variable u in $\vec{u}_2$ is labeled with Δ if $\nu(u) \in B_\infty \setminus (\bigcup_{i=1}^\infty Y_i)$, and with Σ otherwise. In order to choose the linear ordering on the shared variables, we partition the range B_∞ of ν as follows:

$$B_0,\ \ Y_1,\ \ B_1 \setminus (B_0 \cup Y_1),\ \ Y_2,\ \ B_2 \setminus (B_1 \cup Y_2),\ \ Y_3,\ \ B_3 \setminus (B_2 \cup Y_3),\ \ldots$$

Now, let $\vec{v}_1, \vec{w}_1, \ldots, \vec{v}_k, \vec{w}_k$ be a re-ordering of the tuple $\vec{u}_2$ such that the following holds:

1. The tuple $\vec{v}_1$ contains exactly the shared variables whose ν-images are in B_0.
2. For all $i, 1 \le i \le k$, the tuple $\vec{w}_i$ contains exactly the shared variables whose ν-images are in Y_i.
3. For all $i, 1 < i \le k$, the tuple $\vec{v}_i$ contains exactly the shared variables whose ν-images are in $B_{i-1} \setminus (B_{i-2} \cup Y_{i-1})$.

Obviously, this implies that the variables in the tuples $\vec{w}_i$ have label Σ, whereas the variables in the tuples $\vec{v}_i$ have label Δ. Note that some of these tuples may be of dimension 0. The re-ordering determines the linear ordering we choose in Step 4. Let

$$\begin{aligned}\alpha &= \forall\vec{v}_1\exists\vec{w}_1\ldots\forall\vec{v}_k\exists\vec{w}_k\exists\vec{u}_{1,\Sigma}\ \gamma_{2,\Sigma}\\ \beta &= \exists\vec{v}_1\forall\vec{w}_1\ldots\exists\vec{v}_k\forall\vec{w}_k\exists\vec{u}_{1,\Delta}\ \gamma_{2,\Delta}\end{aligned}$$

be the output pair that is obtained by these choices. Let $\vec{y}_i := \nu(\vec{w}_i) \in \vec{Y}$ and $\vec{b}_i := \nu(\vec{v}_i) \in \vec{B}_\infty$. We claim that the sequence $\vec{b}_1, \vec{y}_1, \ldots, \vec{b}_k, \vec{y}_k$ satisfies Condition 2 of Lemma 4.14 for $\varphi = \exists\vec{u}_{1,\Delta}\ \gamma_{2,\Delta}$ and $\mathcal{B}^\Delta_\infty$.[8]

Part (a) of this condition is satisfied since $\mathcal{B}^{\Sigma\cup\Delta}_\infty \models \exists\vec{u}_{1,\Delta}\ \gamma_{2,\Delta}(\nu(\vec{u}_2), \vec{u}_{1,\Delta})$, and thus

$$\mathcal{B}^\Delta_\infty \models \exists\vec{u}_{1,\Delta}\ \gamma_{2,\Delta}(\vec{b}_1, \vec{y}_1, \ldots, \vec{b}_k, \vec{y}_k, \vec{u}_{1,\Delta}).$$

Part (b) of the condition is satisfied since the ν-images of all shared variables in $\vec{u}_2$ are distinct according to our choice in the variable identification step. Finally, part (c) is satisfied because of our choice of the linear ordering. In fact, any component b of $\vec{b}_j$ belongs to B_{j-1}, and is thus an element of $SH^{\mathcal{B}_\infty}_{\mathcal{N}_\infty}(\bigcup_{i=0}^{j-1} Y_i)^\Delta$. For this reason, $Stab_{\mathcal{N}_\infty}(\vec{b}_j) \subseteq \bigcup_{i=0}^{j-1} Y_i$, whereas the components of $\vec{y}_j$ are in Y_j. Thus, the components of $\vec{y}_j$ are not contained in $Stab_{\mathcal{N}_\infty}(\vec{b}_1) \cup \ldots \cup Stab_{\mathcal{N}_\infty}(\vec{b}_{j-1}) \subseteq \bigcup_{i=0}^{j-1} Y_i$.

This shows that we can apply Lemma 4.14, which yields $\mathcal{B}^\Delta \simeq \mathcal{B}^\Delta_\infty \models \beta$. In order to show $\mathcal{A}^\Sigma \models \alpha$, we use the fact that $h_\infty : \mathcal{B}^{\Sigma\cup\Delta} \to \mathcal{A}^{\Sigma\cup\Delta}$ is a $(\Sigma \cup \Delta)$-isomorphism. Thus, $\mathcal{B}^{\Sigma\cup\Delta}_\infty \models \exists\vec{u}_{1,\Sigma}\ \gamma_{2,\Sigma}(\nu(\vec{u}_2), \vec{u}_{1,\Sigma})$ implies that $\mathcal{A}^\Sigma_\infty \models \exists\vec{u}_{1,\Sigma}\ \gamma_{2,\Sigma}(h_\infty(\nu(\vec{u}_2)), \vec{u}_{1,\Sigma})$.

Let $\vec{x}_i := h_\infty(\vec{b}_i) = h_\infty(\nu(\vec{v}_i))$ and $\vec{a}_i := h_\infty(\vec{y}_i) = h_\infty(\nu(\vec{w}_i))$ (for $i = 1, \ldots, k$). We claim that the sequence $\vec{x}_1, \vec{a}_1, \ldots, \vec{x}_k, \vec{a}_k$ satisfies Condition 2 of Lemma 4.14 for $\varphi = \exists\vec{u}_{1,\Sigma}\ \gamma_{2,\Sigma}$ and $\mathcal{A}^\Sigma_\infty$.

Obviously, $\mathcal{A}^\Sigma_\infty \models \exists\vec{u}_{1,\Sigma}\ \gamma_{2,\Sigma}(h_\infty(\nu(\vec{u}_2)), \vec{u}_{1,\Sigma})$ implies that part (a) of the condition is satisfied. To see that part (b) is satisfied, recall that, by our choice in the variable identification step, the ν-images of different shared variables in $\vec{u}_2$ are distinct. Since h_∞ is a bijection, this holds for their $(h_\infty \circ \nu)$-images as well.

Part (c) is an easy consequence of the following properties, which in turn are consequences of the definition of the bijection h_∞ and and its inverse g_∞:

1. Since the components of $\vec{b}_1$ are in B_0, we know that the components of $\vec{x}_1$ are in $X_0 \cup X_1$.

[8]Note that, in contrast to the formulation of the lemma, our sequence starts with a tuple of structure elements instead of atoms. The lemma applies nevertheless since in its formulation we did not assume that all tuples have a non-zero dimension.

2. For $1 < i \leq k$, the components of $\vec{b}_i$ are in $B_{i-1} \setminus (B_{i-2} \cup Y_{i-1})$. Thus, the components of $\vec{x}_i$ are in X_i.

3. For $1 \leq i \leq k$, the components of $\vec{y}_i$ are in Y_i. Thus, the components of $\vec{a}_i$ are in $A_{i-1} \setminus (A_{i-2} \cup Y_{i-1})$.

Thus, we can apply Lemma 4.14, and obtain $\mathcal{A}^\Sigma \simeq \mathcal{A}^\Sigma_\infty \models \alpha$. □

8 Combining Constraint Solvers for Strong SC-Structures: The General Positive Case

For strong SC-structures $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ and $(\mathcal{B}^\Delta, \mathcal{N}, X)$, the structure $\mathcal{A}^\Sigma \otimes \mathcal{B}^\Delta$ is the free amalgamated product of $\mathcal{A}^\Sigma$ and $\mathcal{B}^\Delta$ over X with respect to $Adm(\mathcal{A}^\Sigma, \mathcal{B}^\Delta)$. In this case, our combination method is not restricted to *existential* positive sentences. The main idea is to transform positive sentences (with arbitrary quantifier prefix) into existential positive sentences by Skolemizing the universally quantified variables. In principle, the decomposition algorithm for positive sentences is now applied twice to decompose the input sentence into three positive sentences α, β, ρ, whose validity must respectively be decided in $\mathcal{A}^\Sigma$, $\mathcal{B}^\Delta$, and the absolutely free term algebra over the Skolem functions (see Algorithm 2 below). The restriction to strong SC-structures is necessary since Theorem 6.14 (associativity of free amalgamation) is used in the proof of correctness, and this theorem was proved only for the case of strong SC-structures.

Algorithm 2

The input is a positive sentence φ_1 in the mixed signature $\Sigma \cup \Delta$. We assume that φ_1 is in prenex normalform, and that the matrix of φ_1 is in disjunctive normalform. The algorithm proceeds in two phases.

Phase 1

Via Skolemization of universally quantified variables,[9] φ_1 is transformed into an existential sentence φ_1' over the signature $\Sigma \cup \Delta \cup \Gamma_1$. Here Γ_1 is the

[9] We are Skolemizing *universally* quantified variables since we are interested in validity of the sentence and not in satisfiability.

signature consisting of all the new Skolem function symbols that have been introduced.

Suppose that φ_1' is of the form $\exists \vec{u}_1(\bigvee \gamma_{1,i})$, where the $\gamma_{1,i}$ are conjunctions of atomic formulae. Obviously, φ_1' is equivalent to $\bigvee(\exists \vec{u}_1\ \gamma_{1,i})$, and thus it is sufficient to decide validity of the sentences $\exists \vec{u}_1\ \gamma_{1,i}$. Each of these sentences is used as input for the decomposition algorithm.

The atomic formulae in $\gamma_{1,i}$ may contain symbols from the two (disjoint) signatures Σ and $\Delta \cup \Gamma_1$. In Phase 1 we treat the sentences $\exists \vec{u}_1 \gamma_{1,i}$ by means of Steps 1–4 of the decomposition algorithm, finally splitting them into positive Σ-sentences α and positive $(\Delta \cup \Gamma_1)$-sentences φ_2. Thus, the output of Phase 1 is a finite set of pairs (α, φ_2).

Phase 2

In the second phase, φ_2 is treated exactly as φ_1 was treated before, applying Skolemization to universally quantified variables and Steps 1–4 of the decomposition algorithm a second time. Now we consider the two (disjoint) signatures Δ and $\Gamma = \Gamma_1 \cup \Gamma_2$, where Γ_2 contains the Skolem functions that are introduced by the Skolemization step of Phase 2. We obtain output pairs of the form (β, ρ), where β is a positive sentence over the signature Δ and ρ is a positive sentence over the signature Γ. Together with the corresponding sentence α (over the signature Σ) we thus obtain triples (α, β, ρ) as output.

For each of these triple, the sentence α is now tested for validity in $\mathcal{A}^\Sigma$, β is tested for validity in $\mathcal{B}^\Delta$, and ρ is tested for validity in the absolutely free term algebra $\mathcal{T}(\Gamma, X)$ with countably many generators X, i.e., the free algebra over X for the class of all Γ-algebras.[10] We have seen that this structure is a strong SC-structure with atom set X (Examples 4.9 (3)).

Correctness of Algorithm 2

We want to show that the original sentence φ_1 is valid iff for one of the output triples, all three components are valid in the respective structures. The proof depends on the following lemma, which exhibits an interesting connection between Skolemization and free amalgamation with an absolutely free algebra.

[10] Note that Γ contains no predicate symbols.

Lemma 8.1 *Let $\mathcal{A}^\Sigma$ be a strong SC-structure with atom set X, and let γ be a positive Σ-sentence. Suppose that the existential positive sentence γ' is obtained from γ via Skolemization of the universally quantified variables in γ, introducing the set of Skolem function symbols Γ. Let $\mathcal{B}^\Gamma := \mathcal{T}(\Gamma, X)$, and let $\mathcal{A}_\infty^{\Sigma\cup\Gamma}$ be the free amalgamated product of $\mathcal{A}^\Sigma$ and $\mathcal{B}^\Gamma$ as constructed in Section 6. Then $\mathcal{A}^\Sigma \models \gamma$ if, and only if, $\mathcal{A}_\infty^{\Sigma\cup\Gamma} \models \gamma'$.*

Proof. In order to avoid notational overhead, we assume without loss of generality that existential and universal quantifiers alternate in γ,[11] i.e., $\gamma = \forall u_1 \exists v_1 \ldots \forall u_k \exists v_k\ \varphi(u_1, v_1, \ldots, u_k, v_k)$. Skolemization yields the existential formula $\gamma' = \exists v_1 \ldots \exists v_k\ \varphi(f_1, v_1, f_2(v_1), v_2, \ldots, f_k(v_1, \ldots, v_{k-1}), v_k)$. Thus, Γ consists of k distinct new Skolem functions $f_1, f_2, \ldots, f_k$ having the arities $0, 1, \ldots, k-1$, respectively.

First, assume that $\mathcal{A}^\Sigma \models \gamma$. The structures $\mathcal{A}^\Sigma$ and $\mathcal{A}_\infty^\Sigma$ are isomorphic, and thus

$$(*)\quad \mathcal{A}_\infty^\Sigma \models \forall u_1 \exists v_1 \ldots \forall u_k \exists v_k\ \varphi(u_1, v_1, \ldots, u_k, v_k).$$

Suppose that the Skolem symbols $f_1, f_2, \ldots, f_k$ are interpreted by the functions $f_1^{\mathcal{A}_\infty}, \ldots, f_k^{\mathcal{A}_\infty}$ on the carrier A_∞ of $\mathcal{A}_\infty^{\Sigma\cup\Gamma}$. Because of $(*)$ there exists $a_1 \in A_\infty$ such that $\mathcal{A}_\infty^{\Sigma\cup\Gamma} \models \forall u_2 \exists v_2 \ldots \forall u_k \exists v_k\ \varphi(f_1^{\mathcal{A}_\infty}, a_1, u_2, v_2, \ldots, u_k, v_k)$. Iterating this argument, we obtain $a_1, \ldots, a_k \in A_\infty$ such that

$$\mathcal{A}_\infty^{\Sigma\cup\Gamma} \models \varphi(f_1^{\mathcal{A}_\infty}, a_1, f_2^{\mathcal{A}_\infty}(a_1), a_2, \ldots, f_k^{\mathcal{A}_\infty}(a_1, \ldots, a_{k-1}), a_k).$$

This yields

$$\mathcal{A}_\infty^{\Sigma\cup\Gamma} \models \exists v_1 \ldots \exists v_k\ \varphi(f_1, v_1, f_2(v_1), v_2, \ldots, f_k(v_1, \ldots, v_{k-1}), v_k),$$

i.e., $\mathcal{A}_\infty^{\Sigma\cup\Gamma} \models \gamma'$.

For the converse direction, assume that

$$\mathcal{A}_\infty^{\Sigma\cup\Gamma} \models \exists v_1 \ldots \exists v_k\ \varphi(f_1, v_1, f_2(v_1), v_2, \ldots, f_k(v_1, \ldots, v_{k-1}), v_k).$$

There exist $a_1, \ldots, a_k \in A_\infty$ such that

$$(**)\quad \mathcal{A}_\infty^{\Sigma\cup\Gamma} \models \varphi(f_1^{\mathcal{A}_\infty}, a_1, f_2^{\mathcal{A}_\infty}(a_1), a_2, \ldots, f_k^{\mathcal{A}_\infty}(a_1, \ldots, a_{k-1}), a_k),$$

where $f_1^{\mathcal{A}_\infty}, \ldots, f_k^{\mathcal{A}_\infty}$ again denote the functions on A_∞ that interpret the symbols $f_1, \ldots, f_k$.

Our goal is to apply Lemma 4.14. Obviously, $(**)$ shows that the sequence $f_1^{\mathcal{A}_\infty}, a_1, f_2^{\mathcal{A}_\infty}(a_1), a_2, \ldots, f_k^{\mathcal{A}_\infty}(a_1, \ldots, a_{k-1}), a_k$ satisfies part (a) of Condition 2

[11] Obviously one can introduce additional quantifiers over variables not occurring in γ to generate an equivalent formula of this form.

of Lemma 4.14. It remains to be shown that part (b) and (c) are valid as well. The proof will depend on the following four properties, which are an easy consequence of the fact that $\mathcal{B}_\infty^\Gamma$ is an absolutely free Γ-algebra. Note that the carrier of $\mathcal{B}_\infty^\Gamma$ consists of the Γ-terms over the set (of variables) Y_∞, i.e., the symbols f_i interpret themselves.

(p1) Elements of B_∞ of the form $f_i(b_1, \ldots, b_{i-1})$ and $f_j(b'_1, \ldots, b'_{j-1})$ are distinct if $i \neq j$.

(p2) Elements of B_∞ of the form $f_i(b_1, \ldots, b_{i-1})$ are elements of $B_\infty \setminus Y_\infty$.

(p3) If $b \in B_{m+1} \setminus B_m$, then $f_j(\ldots, b, \ldots) \notin B_m \cup Y_{m+1}$.

(p4) Terms $f_j(b_1, \ldots, b_{j-1})$ are distinct from all their arguments b_ν.

Now, (p1) and (p2) can be used to show part (b) of Condition 2 of Lemma 4.14. By definition of the bijections h_∞ and g_∞, the h_∞-image of $B_\infty \setminus Y_\infty$ is in X_∞, and thus $f_i^{\mathcal{A}_\infty}(a_1, \ldots, a_{i-1}) = h_\infty(f_i(g_\infty(a_1), \ldots, g_\infty(a_{i-1}))) \in X_\infty$ by (p2). This shows that the elements $f_i^{\mathcal{A}_\infty}(a_1, \ldots, a_{i-1})$ of the sequence are in fact atoms, i.e., elements of X_∞. All these atoms are different because of (p1). Indeed, since h_∞ is a bijection, (p1) implies

$$\begin{aligned} f_i^{\mathcal{A}_\infty}(a_1, \ldots, a_{i-1}) = h_\infty(f_i(g_\infty(a_1), \ldots, g_\infty(a_{i-1}))) \neq \\ h_\infty(f_j(g_\infty(a_1), \ldots, g_\infty(a_{j-1}))) = f_j^{\mathcal{A}_\infty}(a_1, \ldots, a_{j-1}) \end{aligned}$$

for all $i \neq j$.

To prove (c), we must show that (for all $i, 1 < i \leq k$) $f_i^{\mathcal{A}_\infty}(a_1, \ldots, a_{i-1})$ is not an element of $Stab_{\mathcal{M}_\infty}(a_1) \cup \ldots \cup Stab_{\mathcal{M}_\infty}(a_{i-1})$. Let $b_1, \ldots, b_{i-1}$ be the images of $a_1, \ldots, a_{i-1}$ under the bijection g_∞, and let m be the minimal number such that $\{a_1, \ldots, a_{i-1}\} \subseteq A_m$. Obviously, this implies that $Stab_{\mathcal{M}_\infty}(a_1) \cup \ldots \cup Stab_{\mathcal{M}_\infty}(a_{i-1}) \subseteq \bigcup_{j=0}^m X_j$.

First, we consider the case where the sequence $a_1, \ldots, a_{i-1}$ contains an element $a_j \in A_m \setminus (A_{m-1} \cup X_m)$. Then $b_j = g_\infty(a_j)$ is an element of Y_{m+1}. Property (p3) yields $f_i(b_1, \ldots, b_{i-1}) \notin B_m \cup Y_{m+1}$, and thus $f_i^{\mathcal{A}_\infty}(a_1, \ldots, a_{i-1}) = h_\infty(f_i(b_1, \ldots, b_{i-1})) \notin A_m \cup X_{m+1}$. Hence $f_i^{\mathcal{A}_\infty}(a_1, \ldots, a_{i-1}) \notin \bigcup_{j=0}^m X_j \subseteq A_m \cup X_{m+1}$, and we are done.

Otherwise, the sequence $a_1, \ldots, a_{j-1}$ contains a non-zero number of elements of X_m (these will be called atoms of type 1), and possibly some elements of A_{m-1}. The latter elements are stabilized by atoms in $\bigcup_{j=0}^{m-1} X_j$ (which will be called atoms of type 2). Recall that $g_\infty(X_m) = B_{m-1} \setminus (B_{m-2} \cup Y_{m-1})$. By (p3), $f_i(b_1, \ldots, b_{i-1}) \notin B_{m-2} \cup Y_{m-1}$, and thus $f_i^{\mathcal{A}_\infty}(a_1, \ldots, a_{i-1}) =$

$h_\infty(f_i(b_1,\ldots,b_{i-1})) \notin A_{m-2} \cup X_{m-1}$. This implies that $f_i^{\mathcal{A}_\infty}(a_1,\ldots,a_{i-1})$ is different from all atoms of type 2. In addition, (p4) says that $f_i(b_1,\ldots,b_{i-1})$ is different from all its arguments $b_1,\ldots,b_{i-1}$. Consequently, $f_i^{\mathcal{A}_\infty}(a_1,\ldots,a_{i-1})$ is distinct from all its arguments $a_1,\ldots,a_{i-1}$, and thus from all atoms of type 1. This completes the proof that Condition 2 of Lemma 4.14 is satisfied.

Applying the lemma, we obtain

$$\mathcal{A}_\infty^{\Sigma\cup\Gamma} \models \forall u_1 \exists v_1 \ldots \forall u_k \exists v_k\ \varphi(u_1, v_1, \ldots, u_k, v_k).$$

Since $\gamma = \forall u_1 \exists v_1 \ldots \forall u_k \exists v_k\ \varphi(u_1, v_1, \ldots, u_k, v_k)$ is a pure Σ-formula, and since $\mathcal{A}^\Sigma$ and $\mathcal{A}_\infty^\Sigma$ are isomorphic, this shows $\mathcal{A}^\Sigma \models \gamma$. $\square$

Correctness of Algorithm 2 is an easy consequence of this lemma.

Proposition 8.2 *$\mathcal{A}_\infty^{\Sigma\cup\Delta} \models \varphi_1$ if, and only if, there exists an output triple (α, β, ρ) such that $\mathcal{A}^\Sigma \models \alpha$, $\mathcal{B}^\Delta \models \beta$, and $\mathcal{T}(\Gamma, X) \models \rho$, where Γ consists of the Skolem functions introduced in Phase 1 and 2 of the algorithm.*

Proof. As before, let "$\otimes$" denote the free amalgamated product of two strong SC-structures, as constructed in Section 6.1. Assume that $\mathcal{A}_\infty^{\Sigma\cup\Delta} \simeq \mathcal{A}^\Sigma \otimes \mathcal{B}^\Delta \models \varphi_1$. By Lemma 8.1 and Theorem 6.14, this implies that $(\mathcal{A}^\Sigma \otimes \mathcal{B}^\Delta) \otimes \mathcal{T}(\Gamma_1, X) \simeq \mathcal{A}^\Sigma \otimes (\mathcal{B}^\Delta \otimes \mathcal{T}(\Gamma_1, X)) \models \varphi_1'$, where φ_1' is the formula obtained from φ_1 by Skolemization. Let $\exists \vec{u}_1 \gamma_1$ be one of the disjuncts in φ_1' satisfied by $\mathcal{A}^\Sigma \otimes (\mathcal{B}^\Delta \otimes \mathcal{T}(\Gamma_1, X))$. Since the decomposition algorithm is correct, one of the output pairs (α, φ_2) generated by applying the decomposition algorithm to $\exists \vec{u}_1 \gamma_1$ satisfies $\mathcal{A}^\Sigma \models \alpha$ and $\mathcal{B}^\Delta \otimes \mathcal{T}(\Gamma_1, X) \models \varphi_2$.

We have shown in Proposition 3.6 that $\mathcal{T}(\Gamma_1, X) \otimes \mathcal{T}(\Gamma_2, X) \simeq \mathcal{T}(\Gamma_1 \cup \Gamma_2, X)$. Applying Lemma 8.1 and Theorem 6.14 a second time, we obtain $(\mathcal{B}^\Delta \otimes \mathcal{T}(\Gamma_1, X)) \otimes \mathcal{T}(\Gamma_2, X) \simeq \mathcal{B}^\Delta \otimes \mathcal{T}(\Gamma_1 \cup \Gamma_2, X) \models \varphi_2'$, where φ_2' is the positive existential sentence that is obtained from φ_2 via Skolemization. The decomposition algorithm, applied to φ_2', thus yields an output pair (β, ρ) at the end of Phase 2 such that $\mathcal{B}^\Delta \models \beta$ and $\mathcal{T}(\Gamma_1 \cup \Gamma_2, X) \models \rho$.

It is easy to see that all arguments used during this proof also apply in the other direction. $\square$

The proposition shows that decidability of the positive theory of the free amalgamated product $\mathcal{A}^\Sigma \otimes \mathcal{B}^\Delta$ can be reduced to decidability of the positive theories of $\mathcal{A}^\Sigma$, $\mathcal{B}^\Delta$, and of an absolutely free term algebra $\mathcal{T}(\Gamma, X)$. It is well-known that the whole first-order theory of absolutely free term algebras is decidable [Mal71, Mah88, CL89].

Theorem 8.3 *If* $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ *and* $(\mathcal{B}^\Delta, \mathcal{N}, X)$ *are strong SC-structures over disjoint signatures, then the (full) positive theory of the free amalgamated product* $\mathcal{A}^\Sigma \otimes \mathcal{B}^\Delta$ *is decidable, provided that the positive theories of* $\mathcal{A}^\Sigma$ *and of* $\mathcal{B}^\Delta$ *are decidable.*

In connection with the Theorems 6.14 and 6.6, this provides the basis for constraint solving in the combination of any finite number of strong SC-structures.

Theorem 8.4 *If* $(\mathcal{A}_1^{\Sigma_1}, \mathcal{M}_1, X), \ldots, (\mathcal{A}_n^{\Sigma_n}, \mathcal{M}_n, X)$ *are strong SC-structures over disjoint signatures, then the (full) positive theory of* $\mathcal{A}_1^{\Sigma_1} \otimes \cdots \otimes \mathcal{A}_n^{\Sigma_n}$ *is decidable, provided that the positive theories of all structures* $\mathcal{A}_i^{\Sigma_i}$ *are decidable* $(1 \leq i \leq n)$.

9 Applications

The prerequisite for combining constraint solvers with the help of our decomposition algorithms is that validity of arbitrary positive sentences is decidable in both components (Theorems 7.1 and 8.3). If we leave the realm of free structures, not many results are known that show that the positive theory of a particular SC-structure is decidable. Nevertheless, two SC-structures that we mentioned in our list of examples 4.9 are known to have a decidable full first order theory:

- The first order theory of the algebra of rational trees—like the theory of the algebra of finite trees—is decidable [Mah88].[12]
- The first order theory of the structure of rational feature trees with arity (compare Examples 4.9, (7)) is decidable. The decidability result has been obtained for the ground structure [BaT94] by giving a complete axiomatization. But it is simple to see that all axioms hold in the non-ground structure as well. Thus, ground and non-ground variant are elementary equivalent, which implies that the first order theory of the non-ground structure is decidable, too.

In general, the problem of deciding validity of existential positive sentences and the problem of deciding validity of arbitrary positive sentences in a given

[12]Maher considers ground tree algebras, but over possibly infinite signatures. Therefore his result can be lifted to the non-ground case by treating variables as constants.

structure can be quite different. For the case of SC-structures, however, the following variant of Lemma 4.14 shows that the difference is not drastic.

Lemma 9.1 *Let $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ be an SC-structure such that $SH^{\mathcal{A}}_{\mathcal{M}}(\emptyset) \neq \emptyset$, let*

$$\forall \vec{u}_1 \exists \vec{v}_1 \ldots \forall \vec{u}_k \exists \vec{v}_k \ \varphi(\vec{u}_1, \vec{v}_1, \ldots, \vec{u}_k, \vec{v}_k)$$

be a positive Σ-sentence, and let, for each $i, 1 \leq i \leq k$, $\vec{x}_i$ be an arbitrary (but fixed) sequence of length $|\vec{u}_i|$ of distinct atoms such that distinct sequences $\vec{x}_i$ and $\vec{x}_j$ do not have common elements. Let $X_{1,i}$ denote the set of all atoms occurring in the sequences $\vec{x}_1, \ldots, \vec{x}_i$ $(i = 1, \ldots, k)$. Then the following conditions are equivalent:

1. *$\mathcal{A}^\Sigma \models \forall \vec{u}_1 \exists \vec{v}_1 \ldots \forall \vec{u}_k \exists \vec{v}_k \ \varphi(\vec{u}_1, \vec{v}_1, \ldots, \vec{u}_k, \vec{v}_k)$,*

2. *there exist $\vec{e}_1 \in SH^{\mathcal{A}}_{\mathcal{M}}(X_{1,1}), \ldots, \vec{e}_k \in SH^{\mathcal{A}}_{\mathcal{M}}(X_{1,k})$ such that $\mathcal{A}^\Sigma \models \varphi(\vec{x}_1, \vec{e}_1, \ldots, \vec{x}_k, \vec{e}_k)$.*

Proof. Recall that we do not assume that sequences $\vec{u}_i$ or $\vec{v}_i$ are non-empty. In the present lemma (in contrast to the situation in Lemma 4.14) there is a subtle difference between the case where the quantifier prefix start with a non-empty block of universal quantifiers and the case where the quantifier prefix start with an empty block of universal quantifiers. Here we shall treat the latter case. It is this case where the condition that $SH^{\mathcal{A}}_{\mathcal{M}}(\emptyset) \neq \emptyset$ is needed.

To prove the first direction, assume that

$$\mathcal{A}^\Sigma \models \exists \vec{v}_1 \forall \vec{u}_2 \exists \vec{v}_2 \ldots \forall \vec{u}_k \exists \vec{v}_k \ \varphi(\vec{v}_1, \vec{u}_2, \vec{v}_2, \ldots, \vec{u}_k, \vec{v}_k).$$

Then there exist elements $\vec{c} \in \vec{A}$ such that

$$\mathcal{A}^\Sigma \models \forall \vec{u}_2 \exists \vec{v}_2 \ldots \forall \vec{u}_k \exists \vec{v}_k \ \varphi(\vec{c}_1, \vec{u}_2, \vec{v}_2, \ldots, \vec{u}_k, \vec{v}_k).$$

Since $SH^{\mathcal{A}}_{\mathcal{M}}(\emptyset) \neq \emptyset$ we may apply a surjective endomorphism $m_1 \in \mathcal{M}$ such that all elements in the stabilizer of $\vec{c}_1$ are mapped to $SH^{\mathcal{A}}_{\mathcal{M}}(\emptyset)$. This implies that $\vec{e}_1 := m_1(\vec{c}_1) \subseteq SH^{\mathcal{A}}_{\mathcal{M}}(\emptyset)$ and $Stab^{\mathcal{A}}_{\Sigma}(\vec{e}_1) = \emptyset$. Since m_1 is surjective we have

$$\mathcal{A}^\Sigma \models \forall \vec{u}_2 \exists \vec{v}_2 \ldots \forall \vec{u}_k \exists \vec{v}_k \ \varphi(\vec{e}_1, \vec{u}_2, \vec{v}_2, \ldots, \vec{u}_k, \vec{v}_k),$$

by Lemma 2.1. Hence there are elements $\vec{c}_2 \in \vec{A}$ such that

$$\mathcal{A}^\Sigma \models \forall \vec{u}_3 \exists \vec{v}_3 \ldots \forall \vec{u}_k \exists \vec{v}_k \ \varphi(m(\vec{e}_1), \vec{x}_2, \vec{c}_2, \ldots, \vec{u}_k, \vec{v}_k),$$

where $\vec{x}_2$ is the sequence given in the lemma. We apply a second surjective endomorphism $m_2 \in \mathcal{M}$ such that all elements in the stabilizer of $\vec{c}_2$ are mapped to $SH^{\mathcal{A}}_{\mathcal{M}}(\vec{x}_2)$. This implies that $\vec{e}_2 := m_2(\vec{c}_2) \subseteq SH^{\mathcal{A}}_{\mathcal{M}}(\vec{x}_2)$ and $Stab^{\mathcal{A}}_{\Sigma}(\vec{e}_2) \subseteq \vec{x}_2$. Since m_2 is surjective we have

$$\mathcal{A}^{\Sigma} \models \forall \vec{u}_3 \exists \vec{v}_3 \ldots \forall \vec{u}_k \exists \vec{v}_k \; \varphi(\vec{e}_1, \vec{x}_2, \vec{e}_2, \vec{u}_3, \vec{v}_3, \ldots, \vec{u}_k, \vec{v}_k),$$

by Lemma 2.1. Proceedings this way, always applying surjective endomorphisms m_i that leave previously selected atoms $\vec{x}_1, \ldots, \vec{x}_{i-1}$ fixed, we obtain the desired sequences $\vec{e}_1 \in SH^{\mathcal{A}}_{\mathcal{M}}(X_{1,1}), \ldots, \vec{e}_k \in SH^{\mathcal{A}}_{\mathcal{M}}(X_{1,k})$ such that $\mathcal{A}^{\Sigma} \models \varphi(\vec{x}_1, \vec{e}_1, \ldots, \vec{x}_k, \vec{e}_k)$.

The converse direction is an immediate consequence of Lemma 4.14. □

Looking at the second condition of the lemma, one sees that a positive sentence can be reduced to an *existential* positive sentence where the universally quantified variables are replaced by atoms (i.e., free constants), and additional restrictions are imposed on the values of the existentially quantified variables. For this reason, it is often not hard to extend decision procedures for the existential positive theory of an SC-structure to a decision procedure for the full positive theory.

In the next two subsections this way of proceeding will be used to prove that the positive theories of the two domains of nested, heriditarily finite wellfounded or non-wellfounded lists (compare Examples 4.9 (6)) are decidable. Similar proofs show that the positive theories of the two domains of nested, heriditarily finite wellfounded or non-wellfounded sets (compare Examples 4.9 (4), (5)) are decidable.

Corollary 9.2 *Simultaneous free amalgamated products have a decidable positive theory if the components are non-ground rational feature structures with arity, finite or rational tree algebras, or nested, heriditarily finite wellfounded or non-wellfounded sets, or nested, heriditarily finite wellfounded or non-wellfounded lists, and if the signatures of the components are disjoint.*

9.1 Nested, hereditarily finite non-wellfounded lists

For the convenience of the reader, let us recall some notation. Let Y denote a countably infinite set of "urelements". The domain $L_{\mathrm{hfnwl}}(Y)$ of nested, hereditarily finite non-wellfounded lists over Y contains all ordered, rational[13] trees where the topmost node has label "$\langle\ \rangle$" (representing a list constructor

[13] A finite or infinite tree is rational if it has only a finite number of distinct substrees.

of variable finite arity), each node that has at least one successor has label "$\langle\ \rangle$", and leaves have label $y \in Y$ or "$\langle\ \rangle$". Let $X = \{\langle y \rangle; y \in Y\}$ denote the atom set. As operations we consider concatenation "$\circ$" and (singleton-) list construction $\langle\cdot\rangle : l \mapsto \langle l \rangle$. Accordingly, formulas are built over the signature $\Sigma := \{\circ, \langle\cdot\rangle\}$. Expressions $\langle\cdot\rangle(t)$ will be written in the form $\langle t \rangle$, and letters $u, v, w, \ldots$ denote variables of the language.

Lemma 9.3 *Validity of positive sentences over $\mathcal{L}_{\mathrm{hfnwl}}(Y)^\Sigma$ is decidable.*

Proof. Let φ_0 be a positive Σ-sentence. We may assume that φ_0 starts with a mixed quantifier prefix, followed by a quantifier free positive matrix γ_0. In order to decide if φ_0 holds in $\mathcal{L}_{\mathrm{hfnwl}}(Y)^\Sigma$, we shall first compute an equivalent sentence φ_1 where the atomic subformulae have the form $v = l_1 \circ \cdots \circ l_s$ $(s \geq 1)$ where v is a variable and the arguments l_i are either variables or they have the form $\langle w \rangle$, where w is a variable. Obviously, the formula φ_1 may obtained from γ_0 by adding equations $u = l$, where u is a new variable, to the matrix, and adding existential quantifications $\exists u$ immediately in front of the actual quantifier free matrix. Let us assume that φ_1 has the form $\forall \vec{u}_1 \exists \vec{v}_1 \ldots \forall \vec{u}_k \exists \vec{v}_k\ \gamma_1(\vec{u}_1, \vec{v}_1, \ldots, \vec{u}_k, \vec{v}_k)$, where γ_1 is the new quantifier free matrix.

Our next aim is to apply Lemma 9.1. For each i, $1 \leq i \leq k$, let $\vec{x}_i$ be an arbitrary, but fixed sequence of distinct atoms of length $|\vec{u}_i|$, such that distinct sequences $\vec{x}_i$ and $\vec{x}_j$ do not have common elements. Let $X_{1,i}$ denote the set of all atoms occurring in the sequences $\vec{x}_1, \ldots, \vec{x}_i$ $(i = 1, \ldots, k)$. By Lemma 9.1, we have to ask if $\gamma_1(\vec{x}_1, \vec{v}_1, \ldots, \vec{x}_k, \vec{v}_k)$[14] has a solution such that the value of each variable v occurring in $\vec{v}_i$ belongs to the stable hull of $X_{1,i}$. By assumption, $\gamma_1(\vec{x}_1, \vec{v}_1, \ldots, \vec{x}_k, \vec{v}_k)$ is a positive Boolean combination of equations. Thus the new equations have the form $l_0 = l_1 \circ \cdots \circ l_s$ $(s \geq 1)$, where l_0 may be an atom $\langle y \rangle$ or a variable, and the remaining arguments l_i are either variables, atoms, or lists of the form $\langle w \rangle$, where w is a variable or an atom. All atoms are in $X_{1,k}$. Without loss of generality we may assume that γ_1 is just a system (i.e., a conjunction) of equations. To simplify the following arguments we consider an equivalent system γ_2 where each equation has the form $v = l_1 \circ \cdots \circ l_s$ $(s \geq 1)$, where the arguments l_i are variables, or atoms of the form $\langle y \rangle$, or lists of the form $\langle w \rangle$, where w is a variable. As we indicated above, such a system can be reached by introducing new equations. For this purpose, a set $\vec{v}$ of new variables has to be introduced by need.

[14] The informal notation $\gamma_1(\vec{x}_1, \vec{v}_1, \ldots, \vec{x}_k, \vec{v}_k)$ indicates that we fix the evaluation of the variables $\vec{u}_1, \ldots, \vec{u}_k$ by mapping them to $\vec{x}_1, \ldots, \vec{x}_k$. Alternatively, we might think of the $\vec{x}_1, \ldots, \vec{x}_k$ as new constants that are, for simplicity, notationally not distinguished from the corresponding atoms that represent their interpretation.

Let us now assign to each variable v of $\vec{v}_i$ its "set of licensed stabilizers" $D_v := X_{1,i}$. For the remaining variables v occurring in $\delta := \gamma_2$ we define $D_v := X_{1,k}$. We shall now give a non-deterministic algorithm, consisting of two steps.

Algorithm 3

The *input* is a system of equations $\delta(\vec{x}_1, \vec{v}_1, \ldots, \vec{x}_k, \vec{v}_k, \vec{v})$, with given sets of licensed stabilizers D_v, for each variable v occurring in the system. Let $W = \vec{v}_1 \cup \ldots \cup \vec{v}_k \cup v$ be the set of variables occurring in δ.

Step 1: *Non-deterministically identify variables as usual (cf. Algorithm 1 in Subsection 7.1). Let W_0 denote the set of representants. To each representant $v \in W_0$ assign the set $D'_v := \bigcap\{D_u \mid u \in [v]\}$ as its new set of licensed stabilizers. Let δ_0 denote the system that is obtained via variable identification.*

Step 2: *We choose a new set of licensed stabilizers $E_v \subseteq D'_v$, for each $v \in W_0$.*

Step 3: *We introduce a new constant $\dot{w}$ for each $w \in W_0$, and one additional new constant c. In each equation $v = l_1 \circ \cdots \circ l_s$ of δ_0, we replace every argument l_i of the form $\langle w \rangle$ (for $w \in W_0$) by the new argument $l'_i := \langle \dot{w} \rangle$. The arguments l_i of the form $u \in W_0$ or $\langle y \rangle$ (with $y \in Y$) are not modified. To each variable $v \in W_0$, we assign its licensed alphabet*

$$F_v := \{y | \langle y \rangle \in E_v\} \cup \{\dot{w} | E_w \subseteq E_v\} \cup \{c\}.$$

Each resulting system δ_1, with fixed licensed alphabet F_v for each variable $v \in W_0$, is one *output system*.

Each output system can be considered as a system δ_1 of word equations, where for each variable $v \in W_0$ a finite alphabet F_v is specified. In fact, all symbols occurring as *elements* of list expressions in δ_1 are *constants* (of the form $\dot{w}$ or $y \in Y$), and all indecomposable arguments l_i are variables. A solution of such a system is a mapping σ that assigns to each variable $v \in W_0$ a word over its licensed alphabet F_v and solves all equations of δ_1. Solvability of these kind of "constrained" systems of word equations is known to be decidable ([Sc90]). Thus, in order to prove Lemma 9.3 it suffices to show that Algorithm 3 is sound and complete. $\square$

Lemma 9.4 (Completeness of Algorithm 3)
If the input system $\delta(\vec{x}_1, \vec{v}_1, \ldots, \vec{x}_k, \vec{v}_k, \vec{v})$ of Algorithm 3, with given sets of li-

censed stabilizers D_v, has a solution in $\mathcal{L}_{\text{hfnwl}}(Y)^{\Sigma}$, then there exists a solvable output system δ_1.

Proof. Suppose that $\delta(\vec{x}_1, \vec{v}_1, \ldots, \vec{x}_k, \vec{v}_k, \vec{v})$, with given sets of licensed stabilizers D_v, has a solution σ. In Step 1 of Algorithm 3 we identify two variables $v, w \in W$ iff $\sigma(v) = \sigma(w)$. Note that this implies that each representant v is mapped to an element whose stabilizer is a subset of the set D'_v defined in Step 1 of the algorithm. Moreover, all elements of W_0 (the set of representants) are mapped to distinct elements under σ. Moreover, σ solves the system δ_0 with the new sets of licensed stabilizers D'_v.

In Step 2, we assign to each variable $v \in W_0$ the new set of licensed stabilizers $E_v := \{x \in X; x \text{ occurs in } v^\sigma\}$. Since σ solves δ_0, respecting licensed stabilizers, we have $E_v \subseteq D'_v$, for all variables v. Thus our choice is admissible and defines a unique output system δ_1.

The solution σ assigns to each variable $v \in W_0$ a list $v^\sigma = \langle m_1, \ldots, m_k \rangle$. Let us distinguish three types of elements. Elements m_i of type 1 have the form y where $\langle y \rangle \in E_v$. Elements m_i of type 2 are the lists which have the form w^σ, for some variable $w \in W_0$. Note that in this case $E_w \subseteq E_v$ and $\dot{w} \in F_v$, by definition of F_v. Elements of type 3 are lists of another form.

We now define a projection π on lists. The projection acts on the elements, henceforth it commutes with concatenation. In more detail, π leaves each element m_i of the form y (type 1) fixed, maps each element m_i of the form w^σ (type 2) to the constant $m'_i := \dot{w}$, and maps elements m_i of type 3 to the constant $m'_i := c$. Note that π is well-defined since all representants have distinct images under σ, by the choice of the variable identification. Let us now assign to each variable $v \in W_0$ the new value $v^{\sigma'} := \pi(v^\sigma) = \pi(\langle m_1, \ldots, m_k \rangle) = \langle m'_1, \ldots, m'_k \rangle$. We have seen that each letter m'_i is in the licensed alphabet F_v of v.

Consider an equation $v = l_1 \circ \cdots \circ l_s$ of δ_0. Since $v^\sigma = l_1^\sigma \circ \cdots \circ l_s^\sigma$ we have $v^{\sigma'} = \pi(v^\sigma) = \pi(l_1^\sigma) \circ \cdots \circ \pi(l_s^\sigma)$. Let $v = l'_1 \circ \cdots \circ l'_s$ be the corresponding equation of δ_1. In order to prove that σ' solves the equation we show that $\pi(l_i^\sigma) = l_i'^{\sigma'}$, for $1 \leq i \leq s$. If l_i has the form $\langle w \rangle$, then $l_i'^{\sigma'} = l'_i = \langle \dot{w} \rangle = \pi(l_i^\sigma)$. If l_i has the form $\langle y \rangle$, for some urelement y, then $l_i'^{\sigma'} = l'_i = l_i = l_i^\sigma = \pi(l_i^\sigma)$. In the remaining case, $l'_i = l_i = u$ is a variable and $\pi(l_i^\sigma) = \pi(u^\sigma) = u^{\sigma'} = l_i'^{\sigma'}$. Thus σ' is a solution of the constrained output system δ_1. $\square$

Lemma 9.5 (Soundness of Algorithm 3)
If an output system δ_1 of Algorithm 3, with licensed alphabet F_v for each

variable $v \in W_0$, has a solution, then the input system δ has a solution in $\mathcal{L}_{\text{hfnwl}}(Y)^\Sigma$.

Proof. Suppose that σ' is a solution of δ_1 that assigns to each variable $v \in W_0$ a word over $F_v = \{y; \langle y\rangle \in E_v\} \cup \{\dot{w}; E_w \subseteq E_v\} \cup \{c\}$, its licensed alphabet. We show how to find a solution of δ_0, the system reached after Step 1 of the algorithm. It is then trivial to construct a solution of the input system δ.

Let $v = l_1 \circ \cdots \circ l_s$ be an equation of δ_0, let $v = l'_1 \circ \cdots \circ l'_s$ be the corresponding equation of δ_1. We have

$$v^{\sigma'} = {l'_1}^{\sigma'} \circ \cdots \circ {l'_s}^{\sigma'}.$$

In order to find an admissible solution σ of δ_0, we shall give an assignment τ that maps each element of $\{\dot{w}|w \in W_0\} \cup \{c\}$ to an element of $L_{\text{hfnwl}}(Y)$ and leaves urelements $y \in Y$ fixed. The mapping τ will be identified with its homomorphic extension on nested (non-wellfounded) lists with urelements in $Y \cup \{\dot{w}|w \in W_0\} \cup \{c\}$. Thus we obtain $v^{\sigma'\tau} = {l'_1}^{\sigma'\tau} \circ \cdots \circ {l'_s}^{\sigma'\tau}$. Hence, in order to show that $\sigma := \sigma' \circ \tau$ is a solution of δ_0 it suffices to prove (a) that each atom $\langle y\rangle$ occurring in the value w^σ of a variable w is always licensed by D'_w, and (b) that $\langle \dot{w}\rangle^{\sigma'\tau}$ $(= \langle \dot{w}\rangle^\tau) = \langle w\rangle^\sigma$, for all $w \in W_0$.

Let us now start with the definition of τ. Consider the mapping

$$\alpha : \qquad \begin{cases} \dot{w} \mapsto w^{\sigma'} & \text{for } w \in W_0, \\ y \mapsto y & \text{for } y \in Y, \\ c \mapsto \langle\ \rangle & \text{empty list.} \end{cases}$$

We identify α with its homomorphic extension on the set of nested non-wellfounded lists with urelements in $Y \cup \{\dot{w}|w \in W_0\} \cup \{c\}$. Let $n \geq 1$ be a natural number, and suppose that (1) $\langle y\rangle \in E_w$, for all urelements y occurring in $\dot{w}^{\alpha^n}$, and that (2) $E_u \subseteq E_w$ for all dotted variables $\dot{u}$ occurring in $\dot{w}^{\alpha^n}$. We assume that (1) and (2) hold for all $w \in W_0$. From the definition of α and from the choice of the licensed alphabets F_w it follows that (1) and (2) hold for each value $w^{\alpha^{n+1}}$ as well.

It is simple to see that the limit of each sequence $(\dot{w}^{\alpha^n})_{n\geq 1}$ defines a unique non-wellfounded hereditarily finite nested list over the set of urelements Y, which we take to be the value of $\dot{w}$ under τ. Furthermore, we define $c^\tau := c^\alpha = \langle\ \rangle$. Note that (1) and (2) guarantee that $\langle y\rangle \in E_w$, for all urelements y occurring in $\dot{w}^\tau$. If $\dot{w}$ occurs in $v^{\sigma'}$, then $E_w \subseteq E_v \subseteq D'_v$, by definition of F_v. It follows that $\sigma := \sigma' \circ \tau$ assigns licensed values to each variable v. Thus (a) is satisfied. Since $w^\sigma = w^{\sigma'\tau} = \dot{w}^{\alpha\tau} = \dot{w}^\tau$ also (b) is satisfied. $\square$

9.2 Nested, hereditarily finite wellfounded lists

The domain $L_{\mathrm{hfl}}(Y)$ of nested, hereditarily finite wellfounded lists over Y contains all ordered, finite trees where the topmost node has label "$\langle\ \rangle$" (representing a list constructor of variable finite arity), each node that has at least one successor has label "$\langle\ \rangle$", and leaves have label $y \in Y$ or "$\langle\ \rangle$". Atom set X, signature Σ, formulas, and operations (lists construction, concatenation) are as before.

Lemma 9.6 *Validity of positive sentences over $\mathcal{L}_{\mathrm{hfl}}(Y)^{\Sigma}$ is decidable.*

Proof. To prove the lemma, we must show, as before, that it is decidable if a system of equations $\delta(\vec{x}_1, \vec{v}_1, \ldots, \vec{x}_k, \vec{v}_k)$ has a solution in $\mathcal{L}^{\Sigma}_{\mathrm{hfl}}(Y)$ such that the value of each variable v occurring in $\vec{v}_i$ belongs to the stable hull of $X_{1,i}$ (where $X_{1,i}$ denotes the set of all atoms occurring in $\vec{x}_1, \ldots, \vec{x}_i$, for each i, $1 \leq i \leq k$). Equations have the form $v = l_1 \circ \cdots \circ l_s$ $(s \geq 1)$, where the arguments l_i are variables, or atoms of the form $\langle y\rangle$, or lists of the form $\langle w\rangle$, where w is a variable. We assign to each variable v of $\vec{v}_i$ its "set of licensed stabilizers" $D_v := X_{1,i}$.

Algorithm 4

The *input* is the constraint system $\delta(\vec{x}_1, \vec{v}_1, \ldots, \vec{x}_k, \vec{v}_k)$ with given sets of licensed stabilizers D_v, for each variable v occurring in the system. Let W denote the set of variables occurring in δ.

Step 1: *Non-deterministically identify variables as usual (cf. Algorithm 1 in Subsection 7.1). Let W_0 denote the set of representants. To each representant $v \in W_0$ assign $D'_v := \bigcap\{D_u \mid u \in [v]\}$ as its new set of licensed stabilizers. Let δ_0 denote the system that is obtained via variable identification.*

Step 2: *For each $v \in W_0$, choose a new set of licensed stabilizers $E_v \subseteq D'_v$. In addition, choose a partial ordering $<$ on W_0 such that $v < w$ implies $E_v \subseteq E_w$.*

Step 3: *Let c be a new constant. In each equation $v = l_1 \circ \cdots \circ l_k$ of δ_0, replace every element l_i of the form $\langle w\rangle$ by the new element $l'_i := \langle \dot{w}\rangle$, introducing a new constant $\dot{w}$ for each variable $w \in W_0$. The elements l_i of the form $u \in W_0$ or $\langle y\rangle$ (with $y \in Y$) are not modified. The new system δ_1 is a system of word equations. To each variable v, we assign its licensed alphabet $F_v := \{y; \langle y\rangle \in E_v\} \cup \{\dot{w}; w < v\} \cup \{c\}$.*

Each system δ_1, with a fixed licensed alphabet F_v for each variable $v \in W_0$, is one *output system*. Again, the proof of Lemma 9.6 is complete when we show that Algorithm 4 is complete and sound. $\square$

Lemma 9.7 (Completeness of Algorithm 4)
If the input system $\delta(\vec{x}_1, \vec{v}_1, \ldots, \vec{x}_k, \vec{v}_k)$ of Algorithm 4, with given sets D_v, has a solution in $\mathcal{L}_{\text{hfl}}(Y)^\Sigma$, then there exists a solvable output system δ_1.

Proof. Suppose that $\delta(\vec{x}_1, \vec{v}_1, \ldots, \vec{x}_k, \vec{v}_k)$, with given sets D_v, has a solution σ.

In Step 1 of Algorithm 4 we identify two variables $v, w \in W$ iff $\sigma(v) = \sigma(w)$. Note that this implies that each representant v is mapped to an element whose stabilizer is a subset of the set D'_v defined in Step 1 of the algorithm. Moreover, all elements of W_0 (the set of representants) are mapped to distinct elements under σ. Thirdly, σ solves the system δ_0 with the new sets of licensed stabilizers D'_v.

In Step 2 of Algorithm 4 we assign to each variable $v \in W_0$ the new set of licensed stabilizers $E_v := \{x \in X; x \text{ occurs in } v^\sigma\}$. Since σ solves the system δ_0 we have $E_v \subseteq D'_v$, for all variables $v \in W_0$. Furthermore, we define $v < w$ iff v^σ is a proper subtree of w^σ. Obviously, "$<$" is a partial ordering on W_0 and $v < w$ implies that $E_v \subseteq E_w$. Thus our choices are admissible and define a unique output system δ_1 of Algorithm 4.

The solution σ assigns to each variable v a list $v^\sigma = \langle m_1, \ldots, m_k \rangle$. We shall distinguish three types of elements. Elements m_i of type 1 have the form y where $\langle y \rangle \in E_v$. Elements m_j of type 2 are the lists which have the form w^σ, for some variable $w \in W_0$. Note that in this case $E_w \subseteq E_v$ and $w < v$, by definition of $<$. Hence $\dot{w} \in F_v$, by definition of F_v. Moreover w is unique, by the variable identification step. Elements of type 3 are lists of another form. We define a projection π on lists that leaves each element m_i of type 1 fixed, maps each element m_i of the form w^σ (type 2) to the constant $m'_i := \dot{w}$ and maps elements m_i of type 3 to the constant $m'_i := c$. Let us assign to each variable $v \in W_0$ the new value $v^{\sigma'} := \pi(v^\sigma) = \pi(\langle m_1, \ldots, m_k \rangle) = \langle m'_1, \ldots, m'_k \rangle$. We have seen that each letter m'_i is in the licensed alphabet F_v of v.

Consider an equation $v = l_1 \circ \cdots \circ l_s$ of δ_0. We have $v^\sigma = l_1^\sigma \circ \cdots \circ l_s^\sigma$ und thus $v^{\sigma'} = \pi(v^\sigma) = \pi(l_1^\sigma) \circ \cdots \circ \pi(l_s^\sigma)$. Take the corresponding equation $v = l'_1 \circ \cdots \circ l'_s$ of δ_1. In order to prove that σ' solves the latter equation we show that $\pi(l_i^\sigma) = l_i'^{\sigma'}$, for $1 \leq i \leq s$. If l_i has the form $\langle w \rangle$ for some $w \in W_0$, then $l_i'^{\sigma'} = l'_i = \langle \dot{w} \rangle = \pi(l_i^\sigma)$. If l_i has the form $\langle y \rangle$, for some urelement y, then

${l'_i}^{\sigma'} = l'_i = l_i = l_i^\sigma = \pi(l_i^{\sigma'})$. In the remaining case, $l'_i = l_i = u$ is a variable and $\pi(l_i^\sigma) = \pi(u^\sigma) = u^{\sigma'} = {l'_i}^{\sigma'}$. Thus σ' is a solution of the constrained output system. □

Lemma 9.8 (Soundness of Algorithm 4)
If an output system δ_1 of Algorithm 4, with licensed alphabet F_v for each variable v in δ_1, has a solution, then the input system δ has a solution in $\mathcal{L}_{\text{hfl}}(Y)^\Sigma$.

Proof. Suppose that σ' is a solution of δ_1 that assigns to each variable $v \in W_0$ a word over $F_v = \{y; \langle y\rangle \in E_v\} \cup \{\dot{w}; w < v\} \cup \{c\}$, the licensed alphabet. We show that the system δ_0 reached after Step 1 has a solution. It follows immediately that the input system δ has a solution.

Let $v = l_1 \circ \cdots \circ l_s$ be an equation of δ_0, let $v = l'_1 \circ \cdots \circ l'_s$ be the corresponding equation of δ_1. We have

$$v^{\sigma'} = {l'_1}^{\sigma'} \circ \cdots \circ {l'_s}^{\sigma'}.$$

In order to find an admissible solution σ of δ_0, we shall give an assignment τ that maps each element of $\{\dot{w} | w \in W_0\} \cup \{c\}$ to an element of $L_{\text{hfl}}(Y)$ and leaves urelements $y \in Y$ fixed. The mapping τ will be identified with its homomorphic extension on nested wellfounded lists with urelements in $Y \cup \{\dot{w} | w \in W_0\} \cup \{c\}$. Thus we obtain $v^{\sigma'\tau} = {l'_1}^{\sigma'\tau} \circ \cdots \circ {l'_s}^{\sigma'\tau}$. Hence, in order to show that $\sigma := \sigma' \circ \tau$ is a solution of δ_0 it suffices to prove (a) that each stabilizer $\langle y\rangle$ occurring in the value w^σ of a variable $w \in W_0$ is licensed by D'_w, and (b) that $\langle \dot{w}^{\sigma'\tau}\rangle$ $(= \langle \dot{w}^\tau\rangle) = \langle w^\sigma\rangle$, for all $w \in W_0$.

Let $c^\tau := \langle\ \rangle$. The remaining part of the mapping τ will be defined by induction, using the partial ordering $<$ on W_0. Let $\dot{v}$ be a dotted variable, and suppose that τ has been defined for all $\dot{w}$ such that $w < v$. We assume (*) that each atom occurring in $\dot{w}^\tau \in L_{\text{hfl}}(Y)$ belongs to E_w, for all $w < v$. We may now define $\dot{v}^\tau := v^{\sigma'\tau}$. In fact, the definition is well-defined since $w < v$ for all dotted $\dot{w}$ occurring in $v^{\sigma'}$, by definition of F_v. This also shows that condition (*) holds again, by induction hypothesis, since $w < v$ implies $E_w \subseteq E_v$, according to Step 2.

If the atom $\langle y\rangle$ occurs in $w^\sigma = w^{\sigma'\tau}$, then either $\langle y\rangle$ occurs in $w^{\sigma'}$, or $\langle y\rangle$ occurs in a value $\dot{u}^\tau$ for some $u < w$. In the former case we have $\langle y\rangle \in E_w$, since σ' respects the licensed alphabet F_w. In the latter case, condition (*) shows that $\langle y\rangle \in E_u \subseteq E_w$. Thus $\langle y\rangle \in D'_w$, which shows that (a) is satisfied.

Similarly (b) holds since $\langle \dot{w}^{\sigma'\tau}\rangle = \langle \dot{w}^\tau\rangle = \langle w^{\sigma'\tau}\rangle = \langle w^\sigma\rangle$. □

10 Conclusion

This paper should be seen as a first step to provide an abstract framework for the combination of constraint languages and constraint solvers. We have introduced the notion "admissible amalgamated product" in order to capture—in an abstract algebraic setting—our intuition of what a combined solution structure should satisfy. It was shown that in certain cases there exists a canonical structure—called the free amalgamated product—that yields a most general admissible closure of a given amalgamation base.

We have introduced a class of structures—called SC-structures—that are equipped with structural properties that guarantee (1) that a canonical amalgamation construction can be applied to SC-structures over disjoint signatures, and (2) that validity of positive existential formulae in the amalgamated structure obtained by this construction can be reduced to validity of positive formulae in the component structures. For the subclass of strong SC-structures we have obtained stronger results. Interestingly, a very similar class of structures has independently been introduced in [SS88, Wil91] in order to characterize a maximal class of algebras where equation (and constraint) solving essentially behaves like unification.[15]

It is interesting to compare the concrete combined solution domains that can be found in the literature with the combined domains obtained by our amalgamation construction. It turns out that there can be differences if the elements of the components have a tree-like structure that allows for infinite paths (as in the examples of non-wellfounded sets and rational trees). In these cases, frequently a combined solution structure is chosen where an infinite number of "signature changes" may occur when following an infinite path in an element of the combined domain. In contrast, our amalgamation construction yields a combined structure where elements allow for a finite number of signature changes only. This indicates that the free amalgamated product, even if it exists, is not necessarily the only interesting combined domain. It remains to be seen which additional natural ways to combine structures exist, and how different ways of combining structures are formally related.

It should be noted that for most of the results presented in the paper the presence of countably many atoms ("variables") in the structures to be combined is an essential precondition. On the other hand, many constraint-based approaches consider ground structures as solution domains. In most

[15] The notion of an SC-structure can be considered as a sort-free version of the concepts that have been discussed in [SS88, Wil91].

cases, however, a corresponding non-ground structure containing the necessary atoms exists. Thus, our combination method can be applied to these non-ground variants. Of course, the combined structure obtained in this way is again non-ground. For *existential* positive formulae, however, validity in the non-ground combined structure is equivalent to validity in the ground variant of the combined structure.[16] This observation has the following interesting consequence. Even in cases where the (full) positive theory of a ground component structure is undecidable, our combination methods can be applied to show decidability of the existential positive theory even for the *ground* combined structure, provided that the (full) positive theories of the non-ground component structures are decidable. Our remark following Lemma 9.1 shows that decidability of the full positive theory of such a non-ground structure can sometimes be obtained by an easy modification of the decision method for the existential positive case. Free semigroups are an example for this situation: the positive theory of a free semigroup with a finite number $n \geq 2$ of generators is undecidable, whereas the positive theory of the countably generated free semigroup (which corresponds to our non-ground case) is decidable [VaR83].

[16]We assume here that the ground structure is a substructure of the non-ground structure and that "substitution" of ground elements for atoms is homomorphic.

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