A complete axiomatization for $\mathcal{ML}_{\omega_1}$

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Abstract

In this paper we present a completeness theorem for the infinitary modal logic $\mathcal{ML}_{\omega_1}$. The proof is based on the new notion of an *infinitary* modal consistency property.[1]

1 Introduction

One can make out several good reasons why infinitary modal logics should deserve our attention. In the first place, and this does not only apply to the modal case, infinitary logics provide a natural means for overcoming the expressive weakness of the corresponding finite systems. Second, several interesting modal logics may be regarded — *via* suitable translations — as fragments of infinitary modal logics; the most popular ones are certainly *propositional dynamic logic* and the *logic of common knowledge*. There is legitimate hope that a deeper understanding of infinitary logics will lead to important insights into their respective fragments. Third, infinitary modal logics themselves might be analysed as fragments of other logics, namely as fragments of infinitary versions of first-order logic.[2] From a logical point of view these fragments show well-behaviour: quite a few metalogical properties are hereditary from logics to their modal fragments. Last but not least, in a recent book ([2]) J. Barwise and L. Moss have pointed out interesting connections between infinitary modal logics and the theory of non-wellfounded sets.

This paper exclusively deals with the logic $\mathcal{ML}_{\omega_1}$, which is the infinitary extension of propositional modal logic that allows conjunctions and disjunctions over countable sets of formulas. Together with its companion piece [8] the paper provides an analysis of this logic with respect to its most basic metalogical properties. In [8] we proved Craig's interpolation theorem for $\mathcal{ML}_{\omega_1}$ and presented a number of preservation results for certain syntactically specified classes of $\mathcal{ML}_{\omega_1}$-formulas. The present paper is concerned with completeness. We introduce an axiomatic calculus, which forms a natural infinitary extension of Kripke's system K, and show that it is complete with respect to the set of valid $\mathcal{ML}_{\omega_1}$-formulas.

[1] After we had finished this paper we discovered that completeness for $\mathcal{ML}_{\omega_1}$ had already been proved in an article by S. Radev ([6]). Since we regard our proof as slightly more elegant and our style of presentation as more lucid, we decided to send our paper to the press though.

[2] This can be done with the aid of a straightforward adaption of J. van Benthem's standard translation to the infinite; obviously, in the case of $\mathcal{ML}_{\omega_1}$ we get $\mathcal{L}_{\omega_1\omega}$ as the target logic.

The paper is structured as follows. In section 2 we recall some basic notions of the syntax and semantics of $\mathcal{ML}_{\omega_1}$. In section 3 we introduce the axiomatic calculus K_{ω_1} and prove its $\mathcal{ML}_{\omega_1}$-soundness. At the beginning of section 4 we give a brief description of the method of modal consisteny properties due to M. Fitting (see [3]). We then indicate why a straightforward adjustment of this method to $\mathcal{ML}_{\omega_1}$ does not work. The section concludes with an informal sketch of an alternative method more suitable for the purposes of $\mathcal{ML}_{\omega_1}$. This new method is based on the notion of an *infinitary* modal consistency property, which is introduced at the beginning of section 5. The main result of this section provides a model existence theorem regarding this type of consistency property. In section 6 we first show that the set of countable consistent sets of $\mathcal{ML}_{\omega_1}$-formulas forms an infinitary modal consistency property. The completeness theorem for $\mathcal{ML}_{\omega_1}$ is then obtained by an application of this result and the model existence theorem from section 5.

2 Syntax and semantics

For the following we fix a countable set $\mathcal{P} := \{p_n \,|\, n \in \omega\}$ of propositional letters. The set $\mathcal{F}_{\omega_1}$ of infinitary modal formulas (over $\mathcal{P}$) is then defined as the smallest set X such that

$\mathcal{P} \subseteq X$,

if φ is in X, then $\neg\varphi$ is in X,

if Φ is a countable subset of X, then $\bigwedge \Phi$ and $\bigvee \Phi$ are in X,[3]

if φ is in X, then $\Diamond\varphi$ and $\Box\varphi$ are in X.

If Φ contains only two formulas φ and ψ, we usually write $(\varphi\wedge\psi)$ and $(\varphi\vee\psi)$ instead of $\bigwedge\{\varphi,\psi\}$ respectively $\bigvee\{\varphi,\psi\}$. We also use $(\varphi \to \psi)$ and $(\varphi \leftrightarrow \psi)$ as convenient metalinguistic abbreviations. The set of subformulas of a modal formula φ, denoted by $sf(\varphi)$, is defined inductively:

$sf(p_n) := \{p_n\}$, for $n \in \omega$,

$sf(\neg\psi) := sf(\psi) \cup \{\neg\psi\}$,

$sf(\bigwedge \Phi) := \bigcup_{\varphi\in\Phi} sf(\varphi) \cup \{\bigwedge \Phi\}$,

$sf(\bigvee \Phi) := \bigcup_{\varphi\in\Phi} sf(\varphi) \cup \{\bigvee \Phi\}$,

$sf(\Box\varphi) := sf(\varphi) \cup \{\Box\varphi\}$,

$sf(\Diamond\varphi) := sf(\varphi) \cup \{\Diamond\varphi\}$.

[3]A set is called countable if it is finite or of cardinality ω. If Φ is the empty set, $\bigwedge \Phi$ is abbreviated by $\top$ and $\bigvee \Phi$ by $\bot$.

The semantics of $\mathcal{ML}_{\omega_1}$ is based on models of the form $\mathfrak{A} = (A, R^{\mathfrak{A}}, V^{\mathfrak{A}})$, where A is a non-empty set, $R^{\mathfrak{A}}$ a binary relation on A, and $V^{\mathfrak{A}}$ a valuation function from $\mathcal{P}$ into the power set of A. A *pointed* model is a pair $(\mathfrak{A}, a)$ consisting of a model $\mathfrak{A}$ and a distinguished element $a \in A$. The truth of a modal formula in a pointed model is defined in a familiar way:

$(\mathfrak{A}, a) \models p_n :\Leftrightarrow a \in V^{\mathfrak{A}}(p_n)$, for $n \in \omega$,

$(\mathfrak{A}, a) \models \neg\varphi :\Leftrightarrow (\mathfrak{A}, a) \not\models \varphi$,

$(\mathfrak{A}, a) \models \bigwedge \Phi :\Leftrightarrow$ for every $\varphi \in \Phi$: $(\mathfrak{A}, a) \models \varphi$,

$(\mathfrak{A}, a) \models \bigvee \Phi :\Leftrightarrow$ there is a $\varphi \in \Phi$: $(\mathfrak{A}, a) \models \varphi$,

$(\mathfrak{A}, a) \models \Diamond\varphi :\Leftrightarrow \exists a' \in A(R^{\mathfrak{A}}aa' \,\&\, (\mathfrak{A}, a') \models \varphi)$,

$(\mathfrak{A}, a) \models \Box\varphi :\Leftrightarrow \forall a' \in A(R^{\mathfrak{A}}aa' \Rightarrow (\mathfrak{A}, a') \models \varphi)$.

Throughout this paper we make use of a special syntactical operation $\sim$, which is defined as follows:

$\sim p_n := \neg p_n$, for $n \in \omega$,

$\sim (\neg\varphi) := \varphi$,

$\sim (\bigwedge \Phi) := \bigvee\{\neg\varphi \mid \varphi \in \Phi\}$,

$\sim (\bigvee \Phi) := \bigwedge\{\neg\varphi \mid \varphi \in \Phi\}$,

$\sim (\Box\varphi) := \Diamond\neg\varphi$,

$\sim (\Diamond\varphi) := \Box\neg\varphi$.

Roughly speaking, given a modal formula φ, $\sim \varphi$ is obtained from φ by replacing the main operator by its dual and by pushing the negation sign one step inside. This type of operation is often met in the framework of infinitary logic (see [1, 4]). It is easy to verify that $\sim \varphi$ and $\neg\varphi$ are equivalent for each $\varphi \in \mathcal{F}_{\omega_1}$.

The standard definition of the modal degree of a formula has the consequence that for every formula φ, $\neg\varphi$ and $\sim \varphi$ are of the same syntactical complexity. On several occasions it will be useful to have a measure of syntactical complexity with respect to which the degree of $\sim \varphi$ is smaller than the degree of $\neg\varphi$, if φ is non-atomic:

$dg(p_n) := dg(\neg p_n) := 0$, for $n \in \omega$,

$dg(\neg\varphi) := dg(\sim \varphi) + 1$, if $\varphi \notin \mathcal{P}$,

$dg(\bigwedge \Phi) := dg(\bigvee \Phi) := sup\{dg(\varphi) \mid \varphi \in \Phi\} + 1$,

$dg(\Box\varphi) := dg(\Diamond\varphi) := dg(\varphi) + 1$.

3 The calculus K_{ω_1}

In this section we introduce the axiomatic calculus K_{ω_1}. As for the heuristic, finding promising axioms and rules turns out to be an easy exercise: we just have to combine Kripke's system K with the propositional part of Keisler's axiomatization of $\mathcal{L}_{\omega_1\omega}$ (see [4]).

Definition 3.1 The calculus K_{ω_1} is defined by the following axiom schemas and rules:

A1 Each substitution instance of a tautology of boolean logic.

A2 $\neg\varphi \leftrightarrow \sim \varphi$.

A3 $\Diamond \bigvee \Phi \to \bigvee\{\Diamond\varphi \,|\, \varphi \in \Phi\}$.[4]

A4 $\varphi \to \bigvee \Phi$, for $\varphi \in \Phi$.

R1 If φ and $\varphi \to \psi$ are provable, then ψ is provable.

R2 If $\varphi \to \psi$ is provable, then $\Diamond\varphi \to \Diamond\psi$ is provable.

R3 If $\varphi \to \psi$ is provable for each $\varphi \in \Phi$, then $\bigvee \Phi \to \psi$ is provable.

A proof in K_{ω_1} is an α-sequence of $\mathcal{ML}_{\omega_1}$-formulas, with $\alpha < \omega_1$, such that each item of the sequence is an instance of one of the axioms A1 to A4 or is inferred from earlier formulas by one of the rules R1 to R3. A modal formula φ is K_{ω_1}-provable, abbreviated by $\vdash_{K_{\omega_1}} \varphi$, iff there is a proof in K_{ω_1} that has φ as its last item. A countable set Φ of $\mathcal{ML}_{\omega_1}$-formulas is called consistent iff $\neg \bigwedge \Phi$ is not provable in K_{ω_1}.

Lemma 3.2 (Soundness) *For every $\varphi \in \mathcal{F}_{\omega_1}$, if $\vdash_{K_{\omega_1}} \varphi$ then $(\mathfrak{A}, a) \models \varphi$ for every pointed model $(\mathfrak{A}, a)$.*

Proof: By transfinite induction on the length of K_{ω_1}-proofs. $\square$

4 Modal consistency properties

A non-empty set S of sets of $\mathcal{ML}$-formulas is said to be a *modal consistency property*, if S is closed under subsets and each $s \in S$ satisfies the following conditions:

c1 if $\varphi \in s$, then $\neg\varphi \notin s$,

c2 if $\neg\varphi \in s$, then $s \cup \{\sim \varphi\} \in S$,

c3 if $(\varphi \wedge \psi) \in s$, then $s \cup \{\varphi\} \in S$ and $s \cup \{\psi\} \in S$,

c4 if $(\varphi \vee \psi) \in s$, then $s \cup \{\varphi\} \in S$ or $s \cup \{\psi\} \in S$,

[4]Important instances of A3 are: $\Diamond(\varphi \vee \psi) \to \Diamond\varphi \vee \Diamond\psi$ and $\Diamond\bot \to \bot$.

c5 if $\Diamond\varphi \in s$, then $\{\varphi\} \cup \{\psi \mid \Box\,\psi \in s\} \in S$.

By utilizing c1 to c5 it can be shown that each member of a modal consistency property S is contained in a saturated theory, that is, for each $s \in S$ there is a set of $\mathcal{ML}$-formulas t such that $s \subseteq t$, and

(i) if $\varphi \in t$, then $\neg\varphi \notin t$,

(ii) if $\neg\varphi \in t$, then $\sim \varphi \in t$,

(iii) if $(\varphi \wedge \psi) \in t$, then $\varphi \in t$ and $\psi \in t$,

(iv) if $(\varphi \vee \psi) \in t$, then $\varphi \in t$ or $\psi \in t$.

If every member of S can be extended to a saturated theory t which also satisfies (v), then S is called a *strong* modal consistency property:

(v) if $\Diamond\varphi \in t$, then $\{\varphi\} \cup \{\psi \mid \Box\,\psi \in t\} \in S$.

What makes a consistency property S a valuable metalogical tool is the fact that we can prove a model existence theorem with respect to it, that is, we can show that each member of S has a model. A careful examination of the proof of this result leads to the insight that in the case of modal logic the proof can only be carried through under the assumption that S is *strong*. To see this, let's recapitulate the main steps of the proof:

(1) The canonical S-model $\mathfrak{A}_S = (A_S, R_S, V_S)$ is defined as follows: Let A_S be the set of saturated theories satisfying (v), put $R_S tt'$ iff $\{\psi \mid \Box\,\psi \in t\} \subseteq t'$, and $V_S(p_n) := \{t \in A_S \mid p_n \in t\}$, for $n \in \omega$.

(2) The following one-way version of the truth lemma is proved by induction: For every $t \in A_S$ and every $\mathcal{ML}$-formula φ, if $\varphi \in t$ then $(\mathfrak{A}_S, t) \models \varphi$. Consider the case $\varphi \doteq \Diamond\psi \in t$. Suppose there is a $t' \in A_S$ with $\psi \in t'$ and $R_S tt'$, then, by induction hypothesis, $(\mathfrak{A}_S, t') \models \psi$, hence $(\mathfrak{A}_S, t) \models \varphi$. Moreover, such t' exists only if the set $s := \{\psi\} \cup \{\chi \mid \Box\,\chi \in t\}$ is contained in an element of A_S; that's the point where the strongness assumption on S must be brought into play. Since t satisfies (v), $s \in S$, thus, by the strongness of S, there is a $t' \in A_S$ such that $s \subseteq t'$, which completes the proof.

(3) The model existence theorem is easily obtained from (2): Suppose $s \in S$, then s can be extended to a saturated theory t. Since S is strong, t can be chosen from A_S, hence, by (2), each element of t is true in $(\mathfrak{A}_S, t)$, thus s is satisfiable.

To prove completeness for $\mathcal{ML}$ it now suffices to show that the set U of all K-consistent sets of $\mathcal{ML}$-formulas forms a *strong* modal consistency property. Suppose this has been proved, then we can reason as follows: Assume $\nvdash_K \varphi$, hence $\{\neg\varphi\}$ is K-consistent, hence $\{\neg\varphi\} \in U$. By (3) $\{\neg\varphi\}$ is satisfiable, hence $\not\models \varphi$. That U satisfies c1 to c4 is quite obvious. For c5 we argue as follows: Let $s \in U$ and $\Diamond\varphi \in s$, and assume $\Sigma := \{\varphi\} \cup \{\psi \mid \Box\,\psi \in s\}$ is not in U. Then Σ is not K-consistent, hence there are $\psi_1, \ldots, \psi_n \in \Sigma$ such that $\vdash_K \varphi \to (\neg\psi_1 \vee \ldots \vee \neg\psi_n)$. By an application of R2 we obtain $\vdash_K \Diamond\varphi \to \Diamond(\neg\psi_1 \vee \ldots \vee \neg\psi_n)$, hence, by the finitary version of A3, $\vdash_K \Diamond\varphi \to (\Diamond\neg\psi_1 \vee \ldots \vee \Diamond\neg\psi_n)$. By A2 we get $\vdash_K \Diamond\varphi \to (\neg\Box\psi_1 \vee \ldots \vee \neg\Box\psi_n)$. Because $\Diamond\varphi$ as well as $\Box\psi_1, \ldots, \Box\psi_n$ are contained in s, this contradicts the consistency of s.

To prove the strongness of U, let $s \in U$. A standard construction provides a chain $\langle s_n \mid n \in \omega\rangle$ of members of U, with $s_0 = s$, such that the union $t := \bigcup_{n\in\omega} s_n$ of this chain is a saturated theory. By using compactness it is easy to verify that t is consistent, hence $t \in U$. That t satisfies (v) is then implied by the fact that U satisfies c5.

When we focus our attention on $\mathcal{ML}_{\omega_1}$ we may first get the impression that the above method can be applied here as well. The notion of a modal consistency property, for instance, is adjusted as follows: We demand that the elements of S are countable, and replace c3 and c4 by c3a respectively c4a:

c3a if $\bigwedge\Phi \in s$, then for each $\varphi \in \Phi$, $s \cup \{\varphi\} \in S$,

c4a if $\bigvee\Phi \in s$, then there is a $\varphi \in \Phi$ such that $s \cup \{\varphi\} \in S$.

Accordingly, we may adjust the definition of a saturated theory. So far so good. But when we turn towards the completeness proof we meet with insurmountable difficulties. Proving that K_{ω_1} completely axiomatizes the set of valid $\mathcal{ML}_{\omega_1}$-formulas requires three things: (a) fixing a suitable set S, (b) verifying that S is a consistency property, (c) showing that S is strong. As for (a), let S be the set of all countable consistent subsets of $\mathcal{F}_{\omega_1}$. That S forms a consistency property is easily checked. What remains is (c); but this is exactly the point where we get stuck. To see this, let $s \in S$. Exploiting the countable fragment of $\mathcal{F}_{\omega_1}$ generated by s — for a precise definition see the next section — we can find a saturated theory t which contains s. To be a bit more precise, we can construct a chain $\langle s_n \mid n \in \omega\rangle$ of elements of S, with $s_0 = s$, such that $t := \bigcup_{n\in\omega} s_n$ is a saturated theory.[5] Suppose $\Diamond\varphi \in t$ and $\{\varphi\} \cup \{\psi \mid \Box\psi \in t\} \notin S$. Analogous to the finite case we obtain $\vdash_{K_{\omega_1}} \Diamond\varphi \to \bigvee\{\neg\Box\psi \mid \Box\psi \in t\}$ by modal reasoning. If we could now assume that t were consistent, then we would get a contradiction like in the finite case. Unfortunately, this is exactly what we must not assume. Since $\mathcal{ML}_{\omega_1}$ lacks compactness, S is not closed under the union of ω-chains, and, what is really bad, there is no other strategy for ensuring the consistency of t within sight.

In the remainder of this paper we develop a new method which avoids the above difficulties and by which we will obtain a completeness proof for $\mathcal{ML}_{\omega_1}$ in section 6. The following remarks provide an informal sketch of its main components. They should help the reader to understand the things to come and give him some motivation for working through the tedious formal details.

Let s be a countable consistent set of $\mathcal{ML}_{\omega_1}$-formulas. According to what we have considered so far there is no guarantee that by way of expanding s we finally reach an element of a (canonical) model in which s holds; what we have to do, instead, is to construct a whole model from the bottom up, that is,

[5] In fact, we can do a slightly better job. We can ensure that t is negation complete in the following sense: for each φ contained in the fragment generated by s, either $\varphi \in t$ or $\neg\varphi \in t$. However, the reader will notice that even this stronger feature does not help.

we have to create all the saturated theories which are needed for satisfying s simultaneously.[6]

To see this, suppose we had already completed the construction of the model $\mathfrak{A}$. Similar to the finite case the elements of A are saturated theories, and $R^{\mathfrak{A}}$ and $V^{\mathfrak{A}}$ are defined accordingly. The key to the whole method is the proof of the truth lemma. So let's consider its problematic case $\Diamond\varphi \in t_1$. To obtain $(\mathfrak{A}, t_1) \models \Diamond\varphi$ we need a $t_2 \in A$ containing φ such that $R^{\mathfrak{A}}t_1t_2$, that is $\{\psi \mid \Box\psi \in t_1\} \subseteq t_2$.

That such a t_2 is at our disposal can be secured as follows: By assumption t_1 has been constructed in an inductive process. Thus there is a natural number n such that $\Diamond\varphi$ is already contained in the n-th approximation s_1^n of t_1. The construction is so designed that there is a number $m > n$ such that $\{\varphi\}$ will be added in the m-th step. As $\{\varphi\}$ should eventually be expanded to the wanted t_2, we put $s_2^m := \{\varphi\}$. From now on we must take care that after each step the approximations of t_1 and t_2 are related to each other so as to allow $R^{\mathfrak{A}}t_1t_2$ in the end. That means that for every $\Box\psi \in s_1^m$ the formula ψ has to be added to s_2^m at some point. But that's not sufficient; we also must take care of formulas $\Box\psi$ that will be added to s_1^m in a later step of the construction, that means that whenever a formula $\Box\psi$ enters s_1^k, with $k > m$, there should be a number $l > k$ such that $\psi \in s_2^l$. Obviously, the latter is only possible if $\{\psi\} \cup s_2^k$ is consistent. To ensure this we use the following trick: first note, that on level m, $\Diamond \bigwedge s_2^m \in s_1^m$ holds by assumption. The trick is to preserve this sort of connection throughout the whole process, that is, to ensure that for every $k > m$, $\Diamond \bigwedge s_2^k \in s_1^k$. It should be obvious that in order to carry this out it is not enough to work with pairs of sets of formulas, we have to consider sequences of arbitrary finite length.

Though the foregoing remarks supply sufficient evidence for the claim that in the case of $\mathcal{ML}_{\omega_1}$ it is much more difficult to construct a model for a given consistent set s than it is in the finite case, the reader may still hope that this complication does not affect the notion of a consistency property proper for $\mathcal{ML}_{\omega_1}$. Unfortunately, this hope has to be dashed. The notion of an infinitary modal consistency property as defined in the next section will turn out to be rather elaborated. To make plain that we really are in need of such a strange looking notion, we jump back into the model construction as described above.

Suppose we have already carried out the construction up to stage m, and assume there is a sequence $\langle s_0^m, \dots, s_n^m \rangle$ such that for every $i < n$, $\Diamond \bigwedge s_{i+1}^m \in s_i^m$; a sequence of this sort is said to be S-perfect. Remember that the sets to be constructed should be saturated theories. To make our point, we consider the disjunction case: Suppose s_n^m contains a formula of the shape $\bigvee \Phi$. Then we have to find a set s'_n and a formula $\varphi \in \Phi$ such that $\varphi \in s'_n$ and $s_n^m \subseteq s'_n$. In fact, our job is much more difficult; we also have to find elements $s'_0, \dots, s'_{n-1}$ of S with $s_i^m \subseteq s'_i$, for each $i < n$, such that $\langle s'_0, \dots, s'_n \rangle$ is a S-perfect sequence.

As we see it there is only one way to make sure that this can be done, namely by adding a respective clause to the definition of a consistency property, that

[6] With regard to this aspect our method has a precursor in the area of infinitary intuitionistic logic (see [5]).

is, by requiring something like the following: S is a consistency property only if for every S-perfect sequence $\langle s_0, \ldots, s_n \rangle$,[7]

C4 if $\bigvee \Phi \in s_n$, then there is a $\varphi \in \Phi$ and a S-perfect sequence $\langle s'_0, \ldots, s'_n \rangle$ such that $\varphi \in s'_n$ and for all $i \leq n$, $s_i \subseteq s'_i$.

Admittedly, this does not look very tempting. What one would prefer is a restricted version of C2 — for obvious reasons we call it C4.2 — which is only formulated for S-perfect sequences of length 2, that is for pairs of elements of S, and a theorem that tells us that, if S satisfies C4.2 then S satisfies the whole C4 as well. However, as the following example suggests this is impossible: Let $s_0 := \{\Diamond \Diamond (p \vee q), \Box \Box \neg p\}$, $s_1 := \{\Diamond(p \vee q)\}$ and $s_2 := \{p \vee q\}$. By the choice of these elements $\langle s_0, s_1, s_2 \rangle$ is S-perfect. Now, suppose S contains the sets $\{\Diamond(p \wedge (p \vee q)), \Diamond(p \vee q)\}$ and $\{p, p \vee q\}$, but contains no s with $\{q, p \vee q\} \subseteq s$. It is easy to check that S does not satisfy C4; note that the sequence $\langle s_0, s_1, s_2 \rangle$ has no suitable expansion. On the other hand, the pair $\langle \{\Diamond(p \wedge (p \vee q)), \Diamond(p \vee q), \{p, p \vee q\}\} \rangle$ is a proper extension of $\langle s_1, s_2 \rangle$ regarding C4.2. Any other attempt to restrict the length of the sequences to be considered in C4 by a finite bound can be wrecked by a similar argument. Of course, these considerations are far from a proof but they should help to deliver our notion of an *infinitary* modal consistency property of its artificial character.

5 Infinitary modal consistency properties

In this section we introduce infinitary modal consistency properties and prove a model existence theorem with respect to them. The proof of this theorem relies on the existence of certain countable fragments of $\mathcal{ML}_{\omega_1}$.

Definition 5.1 Let s be a set of $\mathcal{ML}_{\omega_1}$-formulas. The fragment generated by s — $\mathcal{F}(s)$ for short — is defined as the smallest set X such that

i) $s \subseteq X$,

ii) X is closed under subformulas,

iii) X is closed under $\sim, \neg, \Diamond, \Box, \vee$ and $\wedge$.

Note that if s is countable, then $\mathcal{F}(s)$ is countable as well.

Definition 5.2 Let $\langle s_0, \ldots, s_n \rangle$ be a finite sequence of countable sets of $\mathcal{ML}_{\omega_1}$-formulas such that $\Diamond \bigwedge s_{i+1} \in s_i$, for all $i < n$, and let $\varphi \in \mathcal{F}_{\omega_1}$. By induction (up to n) we define a new sequence $\mathcal{E}(s_0, \ldots, s_n, \varphi)$ of the same length which satisfies the following conditions:

for each $i < n$: $\Diamond \bigwedge [\mathcal{E}(s_0, \ldots, s_n, \varphi)]_{i+1} \in [\mathcal{E}(s_0, \ldots, s_n, \varphi)]_i$,[8]

[7] The official clause C4 in Definition 5.5 looks a bit more constructive: employing the function $\mathcal{E}$, the extensions s'_i are defined explicitly.

[8] In general, if σ is a finite sequence of length n, and $i < n$, then $[\sigma]_i$ denotes the $(i+1)$-th item of σ.

for each $i \leq n$: $s_i \subseteq [\mathcal{E}(s_0, \ldots, s_n, \varphi)]_i$, and

$\varphi \in [\mathcal{E}(s_0, \ldots, s_n, \varphi)]_n$.

For the definition of $\mathcal{E}(s_0, \ldots, s_n, \varphi)$ we use the following auxiliary function:

$f(0) := s_n \cup \{\varphi\}$,

$f(i+1) := s_{n-(i+1)} \cup \{\Diamond \bigwedge f(i)\}$, for $i < n$.

Finally, let $\mathcal{E}(s_0, \ldots, s_n, \varphi) := \langle f(n), \ldots, f(0)\rangle$.

Definition 5.3 Let S be a set of countable sets of $\mathcal{ML}_{\omega_1}$-formulas. A finite sequence $\langle s_0, \ldots, s_n\rangle$ of elements of S is called *a perfect S-sequence*, or simply *S-perfect*, iff $\Diamond \bigwedge s_{i+1} \in s_i$, for every $i < n$.

The following lemma is an easy but useful consequence of the preceding definition.

Lemma 5.4 *Let $\langle s_0, \ldots, s_n\rangle$ and $\langle s'_0, \ldots, s'_{n-1}\rangle$ be two S-perfect sequences, and suppose $s_i \subseteq s'_i$ holds for each $i < n$, then $\langle s'_0, \ldots, s'_{n-1}, s_n\rangle$ is S-perfect.*

Proof: By inspection. $\square$

The central notion of this section is introduced in the next definition.

Definition 5.5 A set S of countable sets of $\mathcal{ML}_{\omega_1}$-formulas is called an infinitary modal consistency property iff

C0 $\emptyset \in S$, and for every $s, s' \in S$: if $s \subseteq s'$ and $\varphi \in s'$, then $s \cup \{\varphi\} \in S$,

C1 for every $s \in S$ and $\varphi \in \mathcal{F}_{\omega_1}$: if $\varphi \in s$ then $\neg\varphi \notin s$,

and every perfect S-sequence $\langle s_0, \ldots, s_n\rangle$ satisfies the following conditions:

C2 If $\neg\varphi \in s_n$, then $\mathcal{E}(s_0, \ldots, s_n, \sim \varphi)$ is a S-sequence.

C3 If $\bigwedge \Phi \in s_n$, then $\mathcal{E}(s_0, \ldots, s_n, \varphi)$ is a S-sequence, for every $\varphi \in \Phi$.

C4 If $\bigvee \Phi \in s_n$, then there is at least one $\varphi \in \Phi$ such that $\mathcal{E}(s_0, \ldots, s_n, \varphi)$ is a S-sequence.

C5 If $\Diamond\varphi \in s_n$, then $\langle s_0, \ldots, s_n, \{\varphi\}\rangle$ is a S-sequence.

C6 If $\Box\varphi \in s_{n-1}$, then $\mathcal{E}(s_0, \ldots, s_n, \varphi)$ is a S-sequence.

In C2 to C6 it is only required that the new sequences $\mathcal{E}(\ldots)$ are S-sequences. That they are perfect is an immediate consequence of Definition 5.2.

Theorem 5.6 (Model Existence) *Let S be an infinitary modal consistency property, and suppose $s \in S$. Then s is satisfiable, that is, there is a pointed model $(\mathfrak{A}, a)$ such that $(\mathfrak{A}, a) \models \bigwedge s$.*

Proof: Let $\mathcal{X}$ be the set $\{\langle 0, i_0, i_1, \ldots, i_{m-1}\rangle \mid m \in \omega \,\&\, \forall k < m(i_k \in \omega \setminus \{0\})\}$. The carrier of the model $(\mathfrak{A}, a)$ to be created consists of saturated sets of $\mathcal{ML}_{\omega_1}$-formulas indexed by elements of $\mathcal{X}$. The relation $R^{\mathfrak{A}}$ is defined by $R^{\mathfrak{A}} t_x t_y$ iff there is a $j \in \omega$ with $y = x \circ \langle j\rangle$, and $V^{\mathfrak{A}}(p_n)$ is the set of $t_x \in A$ such that $p_n \in t_x$, for $n \in \omega$. Finally, $\mathfrak{A}$ is tree-like and generated by a, where $s \subseteq a = t_{\langle 0\rangle}$.

The elements of the model will be created inductively in ω stages. Throughout the construction we must take care that on each level n the sets approximating the elements t_x, s_x^n for short, satisfy the following conditions:

E1 For every $x \in \mathcal{X}$, $s_x^n \in S$.

E2 $\{s_x^n | s_x^n \neq \emptyset\}$ is finite.

E3 If $m \leq n$, then $s_x^m \subseteq s_x^n$.

E4 For every $n, j \in \omega$ and $x, y \in \mathcal{X}$: if $s_y^n \neq \emptyset$ and $y = x \circ \langle j\rangle$, then $\Diamond \bigwedge s_y^n \in s_x^n$.

E5 For every $n, k, j \in \omega$ and $x \in \mathcal{X}$: if $s_{x\circ\langle k\rangle}^n \neq \emptyset$ and $j < k$, then $s_{x\circ\langle j\rangle}^n \neq \emptyset$.

It is easy to see that E1 and E4 imply

E6 If $i_0, i_1, \ldots, i_{m-1} \in \omega \setminus \{0\}$ and $s_{\langle 0, i_0, \ldots, i_{m-1}\rangle}^n \neq \emptyset$, then the sequence $\langle s_{\langle 0\rangle}^n, s_{\langle 0, i_0\rangle}^n, \ldots, s_{\langle 0, i_0, \ldots, i_{m-1}\rangle}^n\rangle$ is S-perfect.

Let $l_0, l_1, l_2, \ldots$ be an enumeration of the set $\{\langle x, \varphi\rangle \mid x \in \mathcal{X} \,\&\, \varphi \in \mathcal{F}(s)\}$ so that each element occurs infinitely often. As $\mathcal{F}(s)$ is countable such an enumeration exists.

For the start of the construction, put $s_{\langle 0\rangle}^0 := s$, and $s_x^0 := \emptyset$ for each $x \neq \langle 0\rangle$. Suppose that the sets s_x^n have been constructed so as to satisfy E1 to E5. Consider $l_n = \langle z, \varphi\rangle$ and let $z = \langle 0, i_0, \ldots, i_{m-1}\rangle$. There are two cases to be distinguished. (i) $\varphi \notin s_z^n$: put $s_x^{n+1} := s_x^n$ for each $x \in \mathcal{X}$. (ii) $\varphi \in s_z^n$: here we have to consider a number of subcases depending on the syntactical shape of φ.

$\varphi \in \mathcal{P}$: Let $s_x^{n+1} := s_x^n$ for every $x \in \mathcal{X}$.

$\varphi \doteq \neg\psi$: Define s_x^{n+1} as $[\mathcal{E}(s_{\langle 0\rangle}^n, s_{\langle 0, i_0\rangle}^n, \ldots, s_{\langle 0, i_0, \ldots, i_{m-1}\rangle}^n, \sim\psi)]_0$, if $x = \langle 0\rangle$, $[\mathcal{E}(s_{\langle 0\rangle}^n, s_{\langle 0, i_0\rangle}^n, \ldots, s_{\langle 0, i_0, \ldots, i_{m-1}\rangle}^n, \sim\psi)]_{j+1}$, if there is a $j < m$ such that $x = \langle 0, i_0, \ldots, i_j\rangle$, and s_x^n else.

By induction hypothesis $\langle s_{\langle 0\rangle}^n, s_{\langle 0, i_0\rangle}^n, \ldots, s_{\langle 0, i_0, \ldots, i_{m-1}\rangle}^n\rangle$ is S-perfect. Using C2 (from Definition 5.5) and the induction hypothesis it is easy to see that the new sets s_x^{n+1} satisfy E1 to E5.

$\varphi \doteq \bigwedge \Phi$: If $\Phi \subseteq s_z^n$, define $s_x^{n+1} := s_x^n$, for each $x \in \mathcal{X}$; otherwise choose the first element ψ from a fixed wellorder of Φ with $\psi \notin s_z^n$. The new sets s_x^{n+1} are defined similar to the foregoing case, just replace $\sim\psi$ by ψ. To prove E1 to E5 we make use of C3 and the induction hypothesis.

$\varphi \doteq \bigvee \Phi$: By C4 there is a $\psi \in \Phi$ such that $\mathcal{E}(s_{\langle 0\rangle}^n, s_{\langle 0, i_0\rangle}^n, \ldots, s_{\langle 0, i_0, \ldots, i_{m-1}\rangle}^n, \psi)$ is a S-sequence. The definition of the sets s_x^{n+1} can now be overtaken from the conjunction case.

$\varphi \doteq \Diamond\psi$: Let k be the smallest $j \in \omega \setminus \{0\}$ with $s_{z\circ\langle j\rangle}^n = \emptyset$. We put $s_x^{n+1} := \{\psi\}$, if $x = z \circ \langle k\rangle$, and $s_x^{n+1} := s_x^n$ else. The rest is clear.

$\varphi \doteq \Box\psi$: Consider the set $G(z) := \{j \in \omega \,|\, s^n_{z\circ\langle j\rangle} \neq \emptyset\}$. Again, we have to make a distinction.

(i) $G(z) = \emptyset$: Put $s^{n+1}_x := s^n_x$ for each $x \in \mathcal{X}$.

(ii) $G(z) \neq \emptyset$: By induction hypothesis and E2, $G(z)$ is finite. Hence, by an application of E5, there is a $k \in \omega \setminus \{0\}$ such that $G(z) = \{1, \ldots, k\}$. Moreover, the induction hypothesis implies for every $j \in G(z)$

(*) $\langle s^n_{\langle 0\rangle}, \ldots, s^n_z, s^n_{z\circ\langle j\rangle}\rangle$ is S-perfect.

By induction we define finite sequences σ^j, for $1 \leq j \leq k$, as follows:

$$\sigma^1 := \mathcal{E}(s^n_{\langle 0\rangle}, s^n_{\langle 0,i_0\rangle}, \ldots, s^n_z, s^n_{z\circ\langle 1\rangle}, \psi), \text{ and}$$

$$\sigma^j := \mathcal{E}([\sigma^{j-1}]_0, \ldots, [\sigma^{j-1}]_m, s^n_{z\circ\langle j\rangle}, \psi), \text{ for } 1 < j \leq k.$$

Finally, we define the new sets s^{n+1}_x as

$s^n_{z\circ\langle j\rangle} \cup \{\psi\}$, if there is a $j \in G(z)$ with $x = z \circ \langle j\rangle$,

$[\sigma^k]_0$, if $x = \langle 0\rangle$,

$[\sigma^k]_{l+1}$, if there is a number $l < m$ such that $x = \langle 0, i_0, \ldots, i_l\rangle$, and

s^n_x else.

Using (*), C6 and Lemma 5.4 it is easy to verify that the so defined sets s^{n+1}_x have all the required features.

Suppose that the construction has been carried out for every $n \in \omega$. To finish the construction we set $t_x := \bigcup_{n\in\omega} s^n_x$ for each $x \in \mathcal{X}$. Now, the model $(\mathfrak{A}, a)$ can be defined as follows. As A choose the set $\{t_x \,|\, t_x \neq \emptyset\}$. Define $R^{\mathfrak{A}}$ by $R^{\mathfrak{A}} t_x t_y :\Leftrightarrow \exists j \in \omega (y = x \circ \langle j\rangle)$, and let $V^{\mathfrak{A}}(p_n) := \{t_x \,|\, p_n \in t_x\}$ for $n \in \omega$. Finally, put $a := t_{\langle 0\rangle}$.

The following statement can then be shown by an induction on the degree of φ: $\forall \varphi \in \mathcal{F}(s) \forall x \in \mathcal{X} (\varphi \in t_x \Rightarrow (\mathfrak{A}, t_x) \models \varphi)$.

Suppose $dg(\varphi) = 0$. Then φ is either atomic or the negation of an atomic formula. In the first case the claim follows by the definition of $V^{\mathfrak{A}}$. For the second case let $\varphi \doteq \neg p_m$; hence there is a minimal $k \in \omega$ with $\neg p_m \in s^k_x$. As s^k_x satisfies E1, C1 implies $p_m \notin s^k_x$. Moreover, by the same argument we obtain $p_m \notin s^n_x$ for every $n > k$, hence $p_m \notin t_x$, hence $(\mathfrak{A}, t_x) \models \varphi$ by the definition of $V^{\mathfrak{A}}$.

If $dg(\varphi) > 0$, the argument depends on the form of φ. Suppose $\varphi \doteq \neg\psi$ with $\psi \notin \mathcal{P}$. By construction there is $n \in \omega$ such that $\neg\psi \in s^n_x$. Consider the smallest $m > n$ with $l_m = \langle x, \neg\psi\rangle$. According to the construction we get $\sim \psi \in s^{m+1}_x$, hence $\sim \psi \in t_x$. Then the induction hypothesis yields $(\mathfrak{A}, t_x) \models \sim \psi$ (note that $dg(\sim \varphi) < dg(\neg\varphi)$), thus $(\mathfrak{A}, t_x) \models \neg\psi$. For conjunction and disjunction we can reason in a similar way. The case $\varphi \doteq \Diamond\psi$ is obvious.

For $\varphi \doteq \Box\psi$, assume $R^{\mathfrak{A}} t_x t_y$. Consider the smallest $m \in \omega$ with $s^m_y \neq \emptyset$; then choose the smallest $n > m$ such that $l_n = \langle x, \Box\psi\rangle$. By construction we obtain $\psi \in s^{n+1}_y$, hence $(\mathfrak{A}, t_y) \models \psi$ by induction hypothesis. From this we easily conclude $(\mathfrak{A}, t_x) \models \Box\psi$.

As an instance of the above claim we obtain $(\mathfrak{A}, t_{\langle 0\rangle}) \models \bigwedge t_{\langle 0\rangle}$ and then $(\mathfrak{A}, t_{\langle 0\rangle}) \models \bigwedge s$. Consequently, we have shown that each element of an infinitary modal consistency property has a model. This completes the proof of the theorem. $\square$

6 Completeness

The main task of this section is to prove that the set U of all countable consistent sets of $\mathcal{ML}_{\omega_1}$-formulas is a consistency property in the sense of Definition 5.5. From this result (Theorem 6.5) the completeness theorem for $\mathcal{ML}_{\omega_1}$ can easily be derived by an application of Theorem 5.6. In the proof of Theorem 6.5 we make use of a number of little results which concern the derivability of certain formulas in K_{ω_1}. For the sake of lucidity we state and prove them as seperate lemmas.

Lemma 6.1 *Let $\langle s_0, \ldots, s_n\rangle$ be U-perfect, and $\vdash \bigwedge s_n \to \bigwedge[\mathcal{E}(s_0, \ldots, s_n, \varphi)]_n$. Then for each $i \leq n$ it holds that $\vdash \bigwedge s_i \to \bigwedge[\mathcal{E}(s_0, \ldots, s_n, \varphi)]_i$.*

Proof: By induction on i we show

$$\vdash \bigwedge s_{n-i} \to \bigwedge[\mathcal{E}(s_0, \ldots, s_n, \varphi)]_{n-i}.$$

The case $i = 0$ holds by assumption. Suppose that

$$\vdash \bigwedge s_{n-i} \to \bigwedge[\mathcal{E}(s_0, \ldots, s_n, \varphi)]_{n-i}.$$

Hence by $R2$ we get

$$\vdash \Diamond \bigwedge s_{n-i} \to \Diamond \bigwedge[\mathcal{E}(s_0, \ldots, s_n, \varphi)]_{n-i}.$$

Since $[\mathcal{E}(s_0, \ldots, s_n, \varphi)]_{n-(i+1)}$ is defined as $s_{n-(i+1)} \cup \{\Diamond \bigwedge[\mathcal{E}(s_0, \ldots, s_n, \varphi)]_{n-i}\}$ and $\Diamond \bigwedge s_{n-i} \in s_{n-(i+1)}$ the desired result follows at once. $\square$

Lemma 6.2 *Let $\langle s_0, \ldots, s_n\rangle$ be a sequence of countable sets of $\mathcal{ML}_{\omega_1}$-formulas such that $\Diamond \bigwedge s_{i+1} \in s_i$ for every $i < n$. Suppose there is a $j < n$ such that s_j is consistent. Then for every $k > j$, s_k is consistent.*

Proof: Assume to the contrary that there is a $k > j$ for which s_k is inconsistent; choose k to be minimal. For this k we conclude

$$\vdash \bigwedge s_k \to \bot,$$

hence, again using $R2$,

$$\vdash \Diamond \bigwedge s_k \to \Diamond \bot$$

and by assumption on $\langle s_0, \ldots, s_n\rangle$

$$\vdash \bigwedge s_{k-1} \to \Diamond \bot.$$

As $\neg \Diamond \bot$ is a theorem of K_{ω_1} the latter contradicts the consistency of s_{k-1}. $\square$

Lemma 6.3 *Let $s \in U$ and Φ a countable set of $\mathcal{ML}_{\omega_1}$-formulas. Then $s \cup \{\bigvee \Phi\}$ is consistent if and only if there is a $\varphi \in \Phi$ such that $s \cup \{\varphi\}$ is consistent.*

Proof: For the direction from left to right we use R3; the other direction is proved by an application of $A4$. $\square$

Lemma 6.4 *Let $\langle s_0, \ldots, s_n \rangle$ be a U-perfect sequence, and suppose $\bigvee \Phi \in s_n$. Then for every $i \leq n$:*

$$\vdash \Diamond \bigwedge s_{n-i} \leftrightarrow \bigvee \{\Diamond \bigwedge [\mathcal{E}(s_0, \ldots, s_n, \varphi)]_{n-i} \,|\, \varphi \in \Phi\}.$$

Proof: For $i = 0$ we argue as follows.
$\Leftarrow$: For every $\varphi \in \Phi$ we get

$$\vdash \bigwedge [\mathcal{E}(s_0, \ldots, s_n, \varphi)]_n \to \bigwedge s_n$$

by Definition 5.2. Then R2 yields

$$\vdash \Diamond \bigwedge [\mathcal{E}(s_0, \ldots, s_n, \varphi)]_n \to \Diamond \bigwedge s_n$$

from which we conclude, by R3,

$$\vdash \bigvee \{\Diamond \bigwedge [\mathcal{E}(s_0, \ldots, s_n, \varphi)]_n \,|\, \varphi \in \Phi\} \to \Diamond \bigwedge s_n.$$

$\Rightarrow$: Without loss of generality we can assume that there is a ψ such that $\bigwedge s_n \doteq \psi \wedge \bigvee \Phi$. By A1 and A4 we first obtain

$$\vdash \varphi \to (\psi \to (\bigwedge s_n \wedge \varphi)),$$

and then

$$\vdash \varphi \to (\psi \to (\bigvee \{\bigwedge s_n \wedge \varphi \,|\, \varphi \in \Phi\}))$$

for every $\varphi \in \Phi$. Thus R3 implies

$$\vdash \bigvee \Phi \to (\psi \to \bigvee \{\bigwedge s_n \wedge \varphi \,|\, \varphi \in \Phi\}),$$

from which we conclude

$$\vdash \Diamond \bigwedge s_n \to \Diamond \bigvee \{\bigwedge s_n \wedge \varphi \,|\, \varphi \in \Phi\}$$

by A1 and R2. Finally, an application of A3 leads to

$$\vdash \Diamond \bigwedge s_n \to \bigvee \{\Diamond(\bigwedge s_n \wedge \varphi) \,|\, \varphi \in \Phi\},$$

where the latter is nothing but the desired result

$$\vdash \Diamond \bigwedge s_n \to \bigvee \{\Diamond \bigwedge [\mathcal{E}(s_0, \ldots, s_n, \varphi)]_n \,|\, \varphi \in \Phi\}.$$

For the induction step let $\bigwedge s_{n-(i+1)} \doteq \psi \wedge \Diamond \bigwedge s_{n-i}$; an application of R2 provides

$$\vdash \Diamond \bigwedge s_{n-(i+1)} \leftrightarrow \Diamond(\psi \wedge \Diamond \bigwedge s_{n-i}).$$

By induction hypothesis it follows that

$$\vdash \Diamond \bigwedge s_{n-(i+1)} \leftrightarrow \Diamond(\psi \wedge \bigvee \{\Diamond \bigwedge [\mathcal{E}(s_0, \ldots, s_n, \varphi)]_{n-i} \,|\, \varphi \in \Phi\}).$$

Using A1, A4, R3 and R2 we eventually obtain

$$\vdash \Diamond \bigwedge s_{n-(i+1)} \leftrightarrow \Diamond \bigvee \{\psi \wedge \Diamond \bigwedge [\mathcal{E}(s_0, \ldots, s_n, \varphi)]_{n-i} \mid \varphi \in \Phi\}.$$

Together with $\vdash \Diamond \bigwedge [\mathcal{E}(s_0, \ldots, s_n, \varphi)]_{n-i} \rightarrow \Diamond \bigwedge s_{n-i}$ this implies

$$\vdash \Diamond \bigwedge s_{n-(i+1)} \leftrightarrow \Diamond \bigvee \{\bigwedge s_{n-(i+1)} \wedge \Diamond \bigwedge [\mathcal{E}(s_0, \ldots, s_n, \varphi)]_{n-i} \mid \varphi \in \Phi\},$$

and then, by A3, A4, R2 and R3,

$$\vdash \Diamond \bigwedge s_{n-(i+1)} \leftrightarrow \bigvee \{\Diamond(\bigwedge s_{n-(i+1)} \wedge \Diamond \bigwedge [\mathcal{E}(s_0, \ldots, s_n, \varphi)]_{n-i}) \mid \varphi \in \Phi\}.$$

Taking Definition 5.2 into account we conclude

$$\vdash \Diamond \bigwedge s_{n-(i+1)} \leftrightarrow \bigvee \{\Diamond \bigwedge [\mathcal{E}(s_0, \ldots, s_n, \varphi)]_{n-(i+1)} \mid \varphi \in \Phi\},$$

which completes the proof. □

Theorem 6.5 *U is an infinitary modal consistency property.*

Proof: We have to show that U satisfies C0 to C6 from Definition 5.5. C0 and C1 are obvious from the Definition of U. For the remaining clauses assume that $\langle s_0, \ldots, s_n \rangle$ is U-perfect. In each case it suffices to verify that the items of the respective $\mathcal{E}(s_0, \ldots, s_n, \varphi)$ are consistent.

For C2 we use A2 and Lemma 6.1. C3 is shown by the same lemma and the K_{ω_1}-provability of $\bigwedge \Phi \rightarrow \varphi$, where the latter follows from A1, A2 and A4. The case C5 is obvious; note that from the inconsistency of φ the inconsistency of $\Diamond\varphi$ would follow by R2 and A3. The remaining two cases require a bit more care.

For C4 suppose $\bigvee \Phi \in s_n$. Then Lemma 6.4 implies

$$\vdash \Diamond \bigwedge s_1 \leftrightarrow \bigvee \{\Diamond \bigwedge [\mathcal{E}(s_0, \ldots, s_n, \varphi)]_1 \mid \varphi \in \Phi\}.$$

As s_0 is consistent and $\Diamond \bigwedge s_1 \in s_0$, an application of Lemma 6.3 provides a $\varphi \in \Phi$ such that $[\mathcal{E}(s_0, \ldots, s_n, \varphi)]_0$, that is $s_0 \cup \{\Diamond \bigwedge [\mathcal{E}(s_0, \ldots, s_n, \varphi)]_1\}$, is consistent. Then Lemma 6.2 ensures the consistency of $[\mathcal{E}(s_0, \ldots, s_n, \varphi)]_i$ for each $i \leq n$; thus $\mathcal{E}(s_0, \ldots, s_n, \varphi)$ is a U-sequence.

To prove C6 suppose $\Box\varphi \in s_{n-1}$. It is easy to see that for every $\psi, \chi \in \mathcal{F}_{\omega_1}$

$$\vdash \Diamond\psi \wedge \Box\chi \rightarrow \Diamond(\psi \wedge \chi).$$

On the assumption that $\Diamond \bigwedge s_n \in s_{n-1}$ and $\Box\varphi \in s_{n-1}$ this yields

$$\vdash \bigwedge s_{n-1} \rightarrow \Diamond(\bigwedge s_n \wedge \varphi).$$

Since $[\mathcal{E}(s_0, \ldots, s_n, \varphi)]_{n-1}$ is defined as $s_{n-1} \cup \{\Diamond(\bigwedge s_n \wedge \varphi)\}$ we obtain

$$\vdash \bigwedge s_{n-1} \rightarrow \bigwedge [\mathcal{E}(s_0, \ldots, s_n, \varphi)]_{n-1}.$$

An application of Lemma 6.1 yields

$$\vdash \bigwedge s_i \rightarrow \bigwedge [\mathcal{E}(s_0, \ldots, s_n, \varphi)]_i$$

for each $i < n$. The consistency of $[\mathcal{E}(s_0, \ldots, s_n, \varphi)]_0$ is then a consequence of the consistency of s_0, and this yields, by an application of Lemma 6.2, the consistency of $[\mathcal{E}(s_0, \ldots, s_n, \varphi)]_i$ for every $i > 0$. □

The paper closes with its main result, the completeness theorem for $\mathcal{ML}_{\omega_1}$.

Theorem 6.6 (Completeness) *Let* $\varphi \in \mathcal{F}_{\omega_1}$*, then* $\models \varphi$ *if and only if* $\vdash_{K_{\omega_1}} \varphi$.

Proof: The soundness part was proved in section 3. The other direction is a straightforward consequence of Theorem 6.5 and Theorem 5.6. $\square$

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