On the Combination of Symbolic Constraints, Solution Domains, and Constraint Solvers*

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Abstract. When combining languages for symbolic constraints, one is typically faced with the problem of how to treat "mixed" constraints. The two main problems are (1) how to define a combined solution structure over which these constraints are to be solved, and (2) how to combine the constraint solving methods for pure constraints into one for mixed constraints. The paper introduces the notion of a "free amalgamated product" as a possible solution to the first problem. Subsequently, we define so-called *simply-combinable structures* (SC-structures). For SC-structures over disjoint signatures, a canonical amalgamation construction exists, which for the subclass of *strong* SC-structures yields the free amalgamated product. The combination technique of [BaS92, BaS94a] can be used to combine constraint solvers for (strong) SC-structures over disjoint signatures into a solver for their (free) amalgamated product. In addition to term algebras modulo equational theories, the class of SC-structures contains many solution structures that have been used in constraint logic programming, such as the algebra of rational trees, feature structures, and domains consisting of hereditarily finite (wellfounded or non-wellfounded) nested sets and lists.

1 Introduction

Many CLP dialects, and some of the related formalisms used in computational linguistics, provide for a combination of several "primitive" constraint languages. For example, in Prolog III [Col90], mixed constraints can be used to express lists of rational trees where some nodes can again be lists etc.; Mukai [Muk91] combines rational trees and record structures, and a domain that integrates rational trees and feature structures has been used in [SmT94]; Rounds [Rou88]

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introduces set-valued feature structures that interweave ordinary feature structures and non-wellfounded sets, and many other suggestions for integrating sets into logic programming exist [DOP91, DoR93].

In this paper, we study techniques for combining symbolic constraints from a more general point of view. On the practical side, these considerations may facilitate the design and implementation of new combined constraint languages and solvers. On the theoretical side, we hope to obtain a better understanding of the principles underlying existing combination methods. This should show their essential similarities and differences, and clarify their limitations.

When combining different constraint systems, at least three problems must be solved. The first problem, namely how to define the set of "mixed" constraints, is usually relatively trivial. The two remaining problems—which will be addressed in this paper—are

(1) how to define the *combined solution structure* over which the mixed constraints are to be solved, and
(2) once this combined structure is fixed, how to *combine constraint solvers* for the single languages in order to obtain a constraint solver for the mixed language.

The first part of this paper is concerned with the first aspect. So far, the problem of combining solution domains has not been discussed in a general and systematic way. The reason is that most of the general combination results obtained until now were concerned with cases where the solution structures are defined by logical theories. In this case, the combined structures are defined by the union of the theories. For example, in unification modulo equational theories, the single solution structures are term algebras $\mathcal{T}(\Sigma_1, X)/{=_{E_1}}$ and $\mathcal{T}(\Sigma_2, X)/{=_{E_2}}$ modulo equational theories E_1 and E_2. Thus, the obvious candidate for the combined structure is $\mathcal{T}(\Sigma_1 \cup \Sigma_2, X)/{=_{E_1 \cup E_2}}$, the term algebra modulo the union $E_1 \cup E_2$ of the theories. It is, however, easy to see that feature structures and the "non-wellfounded" solution domains (such as rational trees) mentioned above cannot be described as such quotient term algebras. For this reason, it is not a priori clear whether there is a canonical way of combining such structures. The same problem also arises for other solution domains of symbolic constraints.

As a possible solution to this problem, we introduce the abstract notion of a "free amalgamated product" of two arbitrary structures in Section 3. Whenever the free amalgamated product of two given structures $\mathcal{A}$ and $\mathcal{B}$ exists, it is unique up to isomorphism, and it is the most general element among all structures that can be considered as a reasonable combination of $\mathcal{A}$ and $\mathcal{B}$. For the case of quotient term algebras $\mathcal{T}(\Sigma_1, X)/{=_{E_1}}$ and $\mathcal{T}(\Sigma_2, X)/{=_{E_2}}$, the free amalgamated product yields the combined term algebra $\mathcal{T}(\Sigma_1 \cup \Sigma_2, X)/{=_{E_1 \cup E_2}}$. This shows that it makes sense to propose the free amalgamated product of two solution structures as an adequate combined solution structure.

With respect to the second problem–the problem of combining constraint solvers[3]–rather general results have been obtained for unification in the union of

[3] The problem of combining constraint solvers should not be confused with kind of

equational theories over disjoint signatures [ScS89, Bou90, BaS92]. These results have been generalized to the case of signatures sharing constants [Rin92, KiR94], and to disunification [BaS93]. Prima facie, such an extension of results seems to be mainly an algorithmic problem. The difficulty, one might think, is to find the correct combination method. A closer look at the results reveals, however, that most of the recent combination algorithms use, modulo details, the same transformation steps.[4] In each case, the real problem is to show correctness of the "old" algorithm in the new situation. In [BaS94a] we have tried to isolate the essential algebraic and logical principles that guarantee that the—seemingly universal—combination scheme works. We found a simple and abstract algebraic condition—called combinability—that guarantees correctness of the combination scheme, and allows for a rather simple proof of this fact. In addition, it was shown that this condition characterizes the class of quotient term algebras (i.e., free algebras), or more generally (if additional predicates are present), the class of free structures. In the above mentioned proof, an explicit construction was given that can be used to amalgamate two quotient term algebras over disjoint signatures, and which yields the combined quotient term algebra as result.

In the second part of this paper it is shown that the concept of a combinable structure and the amalgamation construction can considerably be generalized. This yields combination results that apply to most of the structures mentioned above, and which go far beyond the level of quotient term algebras. To this purpose, a weakened notion of "combinability" is introduced (Section 4). Structures that satisfy this weak form of combinability will be called *simply-combinable structures* (SC-structures).[5] The algebra of rational trees [Mah88], feature structures [APS94, SmT94], but also domains over hereditarily finite (wellfounded or non-wellfounded) nested sets and lists turn out to be SC-structures. The main difference between free structures (treated in [BaS94a]) and SC-structures is that free structures are generated by a (countably infinite) set of (free) generators, whereas this need not be the case for SC-structures (e.g., an infinite rational tree is not generated—in the algebraic sense—by its leaf nodes). This difference makes it necessary to give rather involved proofs [BaS94b] for facts that are trivial for the case of free structures. Nevertheless, a variant of the amalgamation construction of [BaS94a] can be used to combine arbitrary SC-structures $\mathcal{A}$ and $\mathcal{B}$ over disjoint signatures Σ and Δ (Section 5). As a Σ-structure (resp. Δ-structure), the amalgam $\mathcal{A} \otimes \mathcal{B}$ is isomorphic to $\mathcal{A}$ (resp. $\mathcal{B}$). Consequently,

combination problem discussed in [NeO79]. In the latter approach, techniques for deciding *validity* of quantifier-free formulas over a mixed logical alphabet are discussed. Thus, variables are implicitly *universally* quantified. Constraint-solvers, in contrast, ask for *satisfiability* of quantifier-free formulas over a *fixed solution domain*. Hence, variables are implicity *existentially* quantified.

[4] Sometimes, additional steps are introduced just to adapt the general scheme to special situations (e.g., [KiR94, BaS93]). For optimization purposes, steps may be applied in different orders, and delay mechanisms are employed (e.g., [Bou90]).

[5] It has turned out that the notion of an SC-structure is closely related to the concept of a "unification algebra" [ScS88], and to the notion of an "instantiation system" [Wil91].

pure Σ-constraints (resp. Δ-constraints) are solvable in $\mathcal{A}$ (resp. $\mathcal{B}$) iff they are solvable in $\mathcal{A} \otimes \mathcal{B}$. If $\mathcal{A}$ and $\mathcal{B}$ belong to the subclass of strong SC-structures, then it can be shown that $\mathcal{A} \otimes \mathcal{B}$ is in fact the free amalgamated product of $\mathcal{A}$ and $\mathcal{B}$ as defined in Section 3. In this case, the amalgamation construction can be applied iteratedly since $\mathcal{A} \otimes \mathcal{B}$ is again a strong SC-structure.

The combination scheme, in the form given in [BaS92, BaS94a], can be used to combine constraint solvers for two arbitrary SC-structures $\mathcal{A}$ and $\mathcal{B}$ over disjoint signatures into a solver for $\mathcal{A} \otimes \mathcal{B}$ (Section 6). In this general setting, we consider *existential positive sentences* as constraints, and the constraint solvers are decision procedures for validity of such formulae in the given solution structure. Thus, decidability of the *existential* positive theory of $\mathcal{A} \otimes \mathcal{B}$ can be reduced to decidability of the positive theories of $\mathcal{A}$ and $\mathcal{B}$. For the case of strong SC-structures $\mathcal{A}$ and $\mathcal{B}$, the combination method can also treat *general positive sentences*. Thus, in this case, decidability of the *full* positive theory of $\mathcal{A} \otimes \mathcal{B}$ can be reduced to decidability of the positive theories of $\mathcal{A}$ and $\mathcal{B}$. As one concrete application we show that validity of positive sentences is decidable in domains that interweave (finite or rational) trees with hereditarily finite (wellfounded or non-wellfounded) sets and lists. For reasons of space limitation, the rather long and technical proofs had to be omitted here. An internal report, providing complete proofs, is available via ftp [BaS94b].

2 Formal Preliminaries

A signature Σ consists of a set Σ_F of function symbols and a disjoint set Σ_P of predicate symbols (not containing "="), each of fixed arity. Atomic Σ-formulae are built with equality "=" or with predicate symbols $p \in \Sigma_P$ as usual. A *positive* Σ-formula has the form $Q_1 u_1 \ldots Q_k u_k \ \varphi$, where $Q_i \in \{\forall, \exists\}$ and φ is a quantifier-free positive matrix, i.e., built from atoms using conjunction and disjunction only. An *existential positive* Σ-formula is a positive formula where the prefix contains only existential quantifiers. Expressions $\mathcal{A}^\Sigma$ ($\mathcal{A}^\Delta$, ...) denote Σ-structures (Δ-structures, ...) over the same carrier set A, and $f_\mathcal{A}$ ($p_\mathcal{A}$) stands for the interpretation of $f \in \Sigma_F$ ($p \in \Sigma_P$) in $\mathcal{A}^\Sigma$. If Δ is a subset of the signature Σ, then any Σ-structure $\mathcal{A}^\Sigma$ can also be considered as a Δ-structure, $\mathcal{A}^\Delta$, by just forgetting about the interpretation of the additional symbols.

Usually, "constraints" are formulae $\varphi(v_1, \ldots, v_n)$ with free variables. The constraint $\varphi(v_1, \ldots, v_n)$ is solvable in $\mathcal{A}^\Sigma$ iff there are $a_1, \ldots, a_n \in A$ such that $\mathcal{A}^\Sigma \models \varphi(a_1, \ldots, a_n)$. Thus, solvability of φ in $\mathcal{A}^\Sigma$ is equivalent to validity of the sentence $\exists v_1 \ldots \exists v_n \ \varphi(v_1, \ldots, v_n)$ in $\mathcal{A}^\Sigma$. In this paper we shall always use this logical point of view. As constraints we consider positive and existential positive sentences. A constraint is "mixed" if it is built over a mixed signature $\Sigma \cup \Delta$.

A Σ-homomorphism is a mapping h between two structures $\mathcal{A}^\Sigma$ and $\mathcal{B}^\Sigma$ such that $h(f_\mathcal{A}(a_1, \ldots, a_n)) = f_\mathcal{B}(h(a_1), \ldots, h(a_n))$ and $p_\mathcal{A}[a_1, \ldots, a_n]$ implies that $p_\mathcal{B}[h(a_1), \ldots, h(a_n)]$ for all $f \in \Sigma_F$, $p \in \Sigma_P$, and $a_1, \ldots, a_n \in A$. Letters $h, g, \ldots$, possibly with subscript, denote homomorphisms. Whenever the signature Σ is not clear from the context, expressions $h^\Sigma, g^\Sigma, \ldots$ will be used.

A Σ-isomorphism is a bijective Σ-homomorphism $h : \mathcal{A}^\Sigma \to \mathcal{B}^\Sigma$ such that $p_\mathcal{A}[a_1, \ldots, a_n]$ if, and only if, $p_\mathcal{B}[h(a_1), \ldots, h(a_n)]$, for all $a_1, \ldots, a_n \in A$. We write $\mathcal{A}^\Sigma \simeq \mathcal{B}^\Sigma$ to indicate that $\mathcal{A}^\Sigma$ and $\mathcal{B}^\Sigma$ are isomorphic. A Σ-endomorphism of $\mathcal{A}^\Sigma$ is a homomorphism $h^\Sigma : \mathcal{A}^\Sigma \to \mathcal{A}^\Sigma$. With $End_\mathcal{A}^\Sigma$ we denote the monoid of all endomorphisms of the Σ-structure $\mathcal{A}^\Sigma$, with composition as operation. The notation $\mathcal{M} \leq End_\mathcal{A}^\Sigma$ expresses that $\mathcal{M}$ is a submonoid of $End_\mathcal{A}^\Sigma$. If $g : A \to B$ and $h : B \to C$ are mappings, then $g \circ h : A \to C$ denotes their composition.

3 Combination of Structures

Suppose that $\mathcal{B}_1^\Sigma$ and $\mathcal{B}_2^\Delta$ are two structures. In this section we shall discuss the following question: What conditions should a $(\Sigma \cup \Delta)$-structure $\mathcal{C}^{\Sigma\cup\Delta}$ satisfy to be called a "combination" of $\mathcal{B}_1^\Sigma$ and $\mathcal{B}_2^\Delta$? The central definition of this section will be obtained after three steps, each introducing a restriction that is motivated by the example of the combination of term algebras modulo equational theories. The structures $\mathcal{B}_1^\Sigma$ and $\mathcal{B}_2^\Delta$ will be called the *components* in the sequel.

Restriction 1: Homomorphisms that "embed" the components into the combined structure must exist. If the components share a common substructure, then the homomorphisms must agree on this substructure.

It would be too restrictive to demand that the components are substructures of the combined structure. For the case of consistent equational theories E, F over disjoint signatures Σ, Δ, there exist injective homomorphisms of $\mathcal{T}(\Sigma, V)/\!=_E$ and $\mathcal{T}(\Delta, V)/\!=_F$ into $\mathcal{T}(\Sigma \cup \Delta, V)/\!=_{E\cup F}$. For non-disjoint signatures, however, these "embeddings" need no longer be 1–1. Note that even for disjoint signatures Σ and Δ there is a common part, namely the trivial structure represented by the set V of variables. Restriction 1 motivates the following definition.

Definition 1. Let Σ and Δ be signatures, let $\Gamma \subseteq \Sigma \cap \Delta$. A triple $(\mathcal{A}^\Gamma, \mathcal{B}_1^\Sigma, \mathcal{B}_2^\Delta)$ with given homomorphic embeddings $h_{A-B_1}^\Gamma : \mathcal{A}^\Gamma \to \mathcal{B}_1^\Sigma$ and $h_{A-B_2}^\Gamma : \mathcal{A}^\Gamma \to \mathcal{B}_2^\Delta$ will be called an *amalgamation base*. The structure $\mathcal{D}^{\Sigma\cup\Delta}$ *closes* the amalgamation base $(\mathcal{A}^\Gamma, \mathcal{B}_1^\Sigma, \mathcal{B}_2^\Delta)$ iff there are homomorphisms $h_{B_1-D}^\Sigma : \mathcal{B}_1^\Sigma \to \mathcal{D}^\Sigma$ and $h_{B_2-D}^\Delta : \mathcal{B}_2^\Delta \to \mathcal{D}^\Delta$ such that $h_{A-B_1}^\Gamma \circ h_{B_1-D}^\Sigma = h_{A-B_2}^\Gamma \circ h_{B_2-D}^\Delta$. We call $(\mathcal{D}^{\Sigma\cup\Delta}, h_{B_1-D}^\Sigma, h_{B_2-D}^\Delta)$ an *amalgamated product* of $(\mathcal{A}^\Gamma, \mathcal{B}_1^\Sigma, \mathcal{B}_2^\Delta)$.

Restriction 2: The combined structure should share "relevant" structural properties with the components.

This principle accounts for the fact that there must be some kind of (logical, algebraic, algorithmic) relationship between the components and the combined structure. In the case of quotient term algebras $\mathcal{T}(\Sigma, V)/_{=_E}$ and $\mathcal{T}(\Delta, V)/_{=_F}$, the combined algebra $\mathcal{T}(\Sigma \cup \Delta, V)/_{=_{E\cup F}}$ satisfies $E \cup F$. In general, we cannot use this as a condition on the structures that close the amalgamation base since $\mathcal{B}_1^\Sigma$ and $\mathcal{B}_2^\Delta$ are not necessarily defined by logical theories. However, for the case of term algebras there is an equivalent algebraic reformulation:

Proposition 2. *For a $(\Sigma \cup \Delta)$-algebra $\mathcal{C}^{\Sigma\cup\Delta}$ and a countably infinite set (of variables) V, the following conditions are equivalent:*

– *The structure $\mathcal{C}^{\Sigma\cup\Delta}$ satisfies all axioms of $E \cup F$.*
– *For every mapping $g_{V-C} : V \to C$ there exist unique homomorphisms $h_E^\Sigma : \mathcal{T}(\Sigma, V)/_{=_E} \to \mathcal{C}^\Sigma$ and $h_F^\Delta : \mathcal{T}(\Delta, V)/_{=_F} \to \mathcal{C}^\Delta$ extending g_{V-C}.*

In Section 5, where we consider amalgamation of a particular type of structures, we shall restrict the admissible structures for closing an amalgamation base $(\mathcal{A}^\Gamma, \mathcal{B}_1^\Sigma, \mathcal{B}_2^\Delta)$ to structures satisfying the second condition of the proposition (with $\mathcal{B}_1^\Sigma, \mathcal{B}_2^\Delta$ in place of the term algebras). In the remainder of this section it is sufficient to assume that some class of admissible structures $Adm(\mathcal{B}_1^\Sigma, \mathcal{B}_2^\Delta)$ for closing the amalgamation base has been fixed.

Definition 3. Let $(\mathcal{A}^\Gamma, \mathcal{B}_1^\Sigma, \mathcal{B}_2^\Delta)$ be an amalgamation base, let $Adm(\mathcal{B}_1^\Sigma, \mathcal{B}_2^\Delta)$ be a class of $(\Sigma\cup\Delta)$-structures, to be called *admissible structures*. An amalgamated product $(\mathcal{D}^{\Sigma\cup\Delta}, h^\Sigma_{B_1-D}, h^\Delta_{B_2-D})$ of $(\mathcal{A}^\Gamma, \mathcal{B}_1^\Sigma, \mathcal{B}_2^\Delta)$ is called *admissible with respect to* $Adm(\mathcal{B}_1^\Sigma, \mathcal{B}_2^\Delta)$ (or simply *admissible*, if the class of admissible structures is clear from the context) iff $\mathcal{D}^{\Sigma\cup\Delta} \in Adm(\mathcal{B}_1^\Sigma, \mathcal{B}_2^\Delta)$.

Restriction 3: Whenever possible, we want to obtain a most general element among all admissible amalgamated products of the components.
In the case of term algebras, the combined algebra $\mathcal{T}(\Sigma\cup\Delta, V)/_{=_{E\cup F}}$ is not just any algebra satisfying $E \cup F$: it is the free algebra.

Definition 4. Let $(\mathcal{A}^\Gamma, \mathcal{B}_1^\Sigma, \mathcal{B}_2^\Delta)$ be an amalgamation base and let $Adm(\mathcal{B}_1^\Sigma, \mathcal{B}_2^\Delta)$ be a class of admissible $(\Sigma \cup \Delta)$-structures. The admissible amalgamated product $(\mathcal{C}^{\Sigma\cup\Delta}, h^\Sigma_{B_1-C}, h^\Delta_{B_2-C})$ of $\mathcal{B}_1^\Sigma$ and $\mathcal{B}_2^\Delta$ over $\mathcal{A}^\Gamma$ is called a *free amalgamated product with respect to* $Adm(\mathcal{B}_1^\Sigma, \mathcal{B}_2^\Delta)$ iff for every admissible amalgamated product $(\mathcal{D}^{\Sigma\cup\Delta}, h^\Sigma_{B_1-D}, h^\Delta_{B_2-D})$ of $\mathcal{B}_1^\Sigma$ and $\mathcal{B}_2^\Delta$ over $\mathcal{A}^\Gamma$ there exists a *unique* homomorphism $h^{\Sigma\cup\Delta}_{C-D} : \mathcal{C}^{\Sigma\cup\Delta} \to \mathcal{D}^{\Sigma\cup\Delta}$ such that $h^\Sigma_{B_1-D} = h^\Sigma_{B_1-C} \circ h^{\Sigma\cup\Delta}_{C-D}$ and $h^\Delta_{B_2-D} = h^\Delta_{B_2-C} \circ h^{\Sigma\cup\Delta}_{C-D}$.

Free amalgamated products need not exist, but if they exist they are unique up to isomorphism.

Theorem 5. *Let $(\mathcal{A}^\Gamma, \mathcal{B}_1^\Sigma, \mathcal{B}_2^\Delta)$ be an amalgamation base with fixed homomorphic embeddings $h^\Gamma_{A-B_1} : \mathcal{A}^\Gamma \to \mathcal{B}_1^\Sigma$ and $h^\Gamma_{A-B_2} : \mathcal{A}^\Gamma \to \mathcal{B}_2^\Delta$. The free amalgamated product of $\mathcal{B}_1^\Sigma$ and $\mathcal{B}_2^\Delta$ over $\mathcal{A}^\Gamma$ with respect to a given class $\mathrm{Adm}(\mathcal{B}_1^\Sigma, \mathcal{B}_2^\Delta)$ is unique up to $(\Sigma \cup \Delta)$-isomorphism.*

In Section 5 we shall give an explicit construction of the free amalgamated product for the class of "strong SC-structures." For our standard example, term algebras modulo equational theories, the free amalgamated product yields the combined quotient term algebra, which shows that the above definition makes sense:

Proposition 6. *Let $\mathcal{B}_1^\Sigma = \mathcal{T}(\Sigma, V)/_{=_E}$ and $\mathcal{B}_2^\Delta = \mathcal{T}(\Delta, V)/_{=_F}$ for consistent equational theories E and F. Let $\mathrm{Adm}(\mathcal{B}_1^\Sigma, \mathcal{B}_2^\Delta)$ be the class of algebras satisfying (one of) the conditions of Proposition 2. For the amalgamation base $(\mathcal{T}(\Sigma \cap \Delta, V), \mathcal{B}_1^\Sigma, \mathcal{B}_2^\Delta)$, the free amalgamated product with respect to $\mathrm{Adm}(\mathcal{B}_1^\Sigma, \mathcal{B}_2^\Delta)$ is isomorphic to the combined algebra $\mathcal{T}(\Sigma \cup \Delta, V)/_{=_{E\cup F}}$.*

Free amalgamation is obviously commutative if the class of admissible structures satisfies $Adm(\mathcal{B}_1^\Sigma, \mathcal{B}_2^\Delta) = Adm(\mathcal{B}_2^\Delta, \mathcal{B}_1^\Sigma)$. Some of our results concerning combination of constraint solvers depend on the assumption that free amalgamation is associative as well. In order to guarantee associativity, some conditions on the classes of admissible structures have to be imposed (see [BaS94b] for details).

It should be noted that notions of "amalgamated product," similar to the one given above, can be found in universal algebra, model theory, and category theory ([Mal73, Che76, DrG93]). There, however, amalgamation is typically studied for structures over the same signature. Moreover, in most cases these structures satisfy a fixed set of axioms (e.g., those for groups, fields, skew fields, etc.).

4 Simply Combinable Structures

In this section we shall introduce the concept of a simply combinable (SC-) structure. This purely algebraic notion yields a large class of structures for which an amalgamated product can be obtained by an explicit construction, provided that the components have disjoint signatures. In this case, general techniques exist that can be used to combine constraint solvers for the components in order to obtain a constraint solver for the amalgamated structure. Many typical domains for constraint-based reasoning turn out to be SC-structures. Quotient term algebras will serve as illustrating and motivating example for the abstract definitions. In the sequel, let $\mathcal{T} := \mathcal{T}(\Sigma_F, V)/_{=_E}$ be such an algebra.

Two endomorphisms of $\mathcal{T}$ that coincide on a set $U \subseteq V$ of variables also coincide on all terms that are built over U. Abstracting this property, we arrive at the following two definitions.

Definition 7. Let A_0, A_1 be subsets of the Σ-structure $\mathcal{A}^\Sigma$, and let $\mathcal{M} \leq End_{\mathcal{A}}^\Sigma$. Then A_0 *stabilizes* A_1 *with respect to* $\mathcal{M}$ iff all elements h_1 and h_2 of $\mathcal{M}$ that coincide on A_0 also coincide on A_1.

The reason for considering submonoids of $End_{\mathcal{A}}^\Sigma$ is that in some cases (such as for feature structures) not all endomorphisms will be of interest in our context. In the sequel, we consider a fixed Σ-structure $\mathcal{A}^\Sigma$; $\mathcal{M}$ always denotes a submonoid of $End_{\mathcal{A}}^\Sigma$.

Definition 8. For $A_0 \subseteq A$ the *stable hull of* A_0 *with respect to* $\mathcal{M}$ is the set

$$SH_{\mathcal{M}}^{\mathcal{A}}(A_0) := \{a \in A;\ A_0 \text{ stabilizes } \{a\} \text{ with respect to } \mathcal{M}\}.$$

The stable hull of a set A_0 has properties that are similar to those of the subalgebra generated by A_0: (1) $SH_{\mathcal{M}}^{\mathcal{A}}(A_0)$ is a Σ-substructure of $\mathcal{A}^\Sigma$, and (2) $A_0 \subseteq SH_{\mathcal{M}}^{\mathcal{A}}(A_0)$. In general, however, the stable hull can be larger than the generated subalgebra.

Definition 9. The set $X \subseteq A$ is an $\mathcal{M}$*-atom set for* $\mathcal{A}^\Sigma$ if every mapping $X \to A$ can be extended to an endomorphism in $\mathcal{M}$. If $\mathcal{M} = End_{\mathcal{A}}^\Sigma$, then X is simply called an *atom set* for $\mathcal{A}^\Sigma$.

For $\mathcal{T}$, the set of variables V is an atom set. Two subalgebras generated by subsets V_0, V_1 of V of the same cardinality are isomorphic. The same holds for atom sets and their stable hulls.

Lemma 10. *Let X_0, X_1 be two $\mathcal{M}$-atom sets of $\mathcal{A}^\Sigma$ of the same cardinality. Then every bijection $h_0 : X_0 \to X_1$ can be extended to an isomorphism between $SH^{\mathcal{A}}_{\mathcal{M}}(X_0)$ and $SH^{\mathcal{A}}_{\mathcal{M}}(X_1)$.*

We are now ready to introduce the main concept of this paper.

Definition 11. A countably infinite Σ-structure $\mathcal{A}^\Sigma$ is an *SC-structure* iff there exists a monoid $\mathcal{M} \leq End^\Sigma_{\mathcal{A}}$ such that $\mathcal{A}^\Sigma$ has an infinite $\mathcal{M}$-atom set X where every $a \in A$ is stabilized by a finite subset of X with respect to $\mathcal{M}$. We denote this SC-structure by $(\mathcal{A}^\Sigma, \mathcal{M}, X)$. If $\mathcal{M} = End^\Sigma_{\mathcal{A}}$, then $(\mathcal{A}^\Sigma, End^\Sigma_{\mathcal{A}}, X)$ is called a *strong SC-structure.*

Examples 12 The following examples show that in fact many solution domains for symbolic constraints are SC-structures.

- Let Σ_F be a finite set of function symbols. The free algebra $\mathcal{T}(\Sigma_F, V)/{=_E}$ modulo the equational theory E with countably infinite generator set V is a strong SC-structure with atom set V. The same holds for free structures, as considered in [BaS94a].
- Let K be a field, let $\Sigma_K := \{+\} \cup \{s_k; k \in K\}$. The K-vector space spanned by a countably infinite basis X is a strong SC-structure over the atom set X. Here "+" is interpreted as addition of vectors, and s_k denotes scalar multiplication with $k \in K$.
- Let Σ_F be a finite set of function symbols, and let $\mathcal{R}^{\Sigma_F}$ be the algebra of rational trees where leaves are labelled with constants from Σ_F or with variables from the countably infinite set (of variables) V. It is easy to see that every mapping $V \to R$ can be extended to a unique endomorphism of $\mathcal{R}^{\Sigma_F}$, and that $(\mathcal{R}^{\Sigma_F}, End^{\Sigma_F}_{\mathcal{R}}, V)$ is a strong SC-structure. Note, however, that $\mathcal{R}^{\Sigma_F}$ is not generated by V.
- Let $V_{\mathrm{hfs}}(Y)$ be the set of all nested, hereditarily finite (standard, i.e., well-founded) sets over the countably infinite set of "urelements" Y. Thus, each set $M \in V_{\mathrm{hfs}}(Y)$ is finite, and the elements of M are either atomic elements in Y or sets in $V_{\mathrm{hfs}}(Y)$, the same holds for elements of elements etc. There are no infinite descending membership sequences. Since union is not defined for the urelements $y \in Y$, the urelements will not be treated as sets here. Let $X := \{\{y\} \mid y \in Y\}$. Let $h : X \to V_{\mathrm{hfs}}(Y)$ be an arbitrary mapping. We want to show that there exists a unique extension of h to a mapping $\hat{h} : V_{\mathrm{hfs}}(Y) \to V_{\mathrm{hfs}}(Y)$ that is homomorphic with respect to union "$\cup$" and set construction $\{\cdot\}$. Each $M \in V_{\mathrm{hfs}}(Y)$ can uniquely be represented in the form $M = x_1 \cup \ldots \cup x_k \cup \{M_1\} \cup \ldots \cup \{M_l\}$ where $x_i \in X$, for $1 \leq i \leq k$, and where the M_i are the elements of M that belong to $V_{\mathrm{hfs}}(Y)$. By induction (on nesting depth), we may assume that $\hat{h}(M_i)$ is already defined $(1 \leq i \leq l)$. Obviously $\hat{h}(M) := h(x_1) \cup \ldots \cup h(x_k) \cup \{\hat{h}(M_1)\} \cup \ldots \cup \{\hat{h}(M_l)\}$ is one and

the only way of extending $\hat{h}$ in a homomorphic way to the set M of deeper nesting. For $M = x \in X$ we obtain $\hat{h}(x) = h(x)$, thus $\hat{h}$ is an extension of h. Moreover, each mapping $\hat{h}$ is in fact homomorphic with respect to union "$\cup$" and set construction "$\{\cdot\}$". It follows easily that $\hat{h}_1 \circ \hat{h}_2$ is the unique extension of $h_1 \circ \hat{h}_2 : X \to V_{\text{hfs}}(Y)$, for all mappings $h_1, h_2 : X \to V_{\text{hfs}}(Y)$, which implies that $\mathcal{M} := \{\hat{h} \mid h : X \to V_{\text{hfs}}(Y)\}$ is closed under composition. Obviously, identity on $V_{\text{hfs}}(Y)$ belongs to $\mathcal{M}$. Thus $V_{\text{hfs}}(Y)$, with union "$\cup$" and set construction "$\{\cdot\}$", is a strong SC-structure with atom set X.

- Similarly it can be seen that the domain of heriditarily finite *non-wellfounded* sets[6] over a countably infinite set of urelements Y, with union "$\cup$" and set construction "$\{\cdot\}$", is a strong SC-structure over the atom set $X = \{\{y\}; y \in Y\}$.
- The two domains of nested, hereditarily finite (1) wellfounded or (2) non-wellfounded lists over the countably infinite set of urelements Y, with concatenation "$\circ$" as binary operation and with list construction $\langle\cdot\rangle : l \mapsto \langle l\rangle$, are strong SC-structures over the atom set $X = \{\langle y\rangle; y \in Y\}$ of all lists with one element $y \in Y$. Formally, these domains can be described as the set of all (1) finite or (2) rational trees where the topmost node has label "$\langle\ \rangle$" (representing a list constructor of varying finite arity), nodes with successors have label "$\langle\ \rangle$", and leaves have labels $y \in Y$ or "$\langle\ \rangle$".
- Let Lab, Fea, and X be mutually disjoint infinite sets of labels, features, and atoms respectively. Following [APS94], a feature tree is a partial function $t : Fea^* \to Lab \cup X$ whose domain is prefix closed (i.e., if $pq \in dom(t)$ then $p \in dom(t)$ for all words $p, q \in Fea^*$), and in which atoms do not label interior nodes (i.e., if $p(t) = x \in X$ then there is no $f \in Fea$ with $pf \in dom(t)$). As usual, *rational* feature trees are required to have only finitely many subtrees. In addition, they must be finitely branching.
  We use the set R of all rational feature trees as carrier set of a structure $\mathcal{R}^\Sigma$ whose signature contains a unary predicate L for every label $L \in Lab$, and a binary predicate f for every $f \in Fea$. The interpretation $L_\mathcal{R}$ of L in $\mathcal{R}$ is the set of all rational feature trees having root label L. The interpretation $f_\mathcal{R}$ of f consists of all pairs $(t_1, t_2) \in R \times R$ such that $t_1(f)$ is defined and t_2 is the subtree of t_1 at f. The structure $\mathcal{R}^\Sigma$ defined this way can be seen as a non-ground version of the solution domain used in [APS94].
  Each mapping $h : X \to R$ has a unique extension to an endomorphism of $\mathcal{R}^\Sigma$ that acts like a substitution, replacing each leaf with label $x \in X$ by the feature tree $h(x)$. With composition, the set of these substitution-like endomorphisms yield a monoid $\mathcal{M}$. Thus $(\mathcal{R}^\Sigma, \mathcal{M}, X)$ is an SC-structure. In this case, we do not have a strong SC-structure since $\mathcal{R}^\Sigma$ has endomorphisms that modify non-leaf nodes (e.g., by introducing new feature-edges for such internal nodes).

[6] Non-wellfounded sets, sometimes called hypersets, became prominent through [Acz88]. They can have infinite descending membership sequences. The heriditarily finite non-wellfounded sets are those having a "finite picture," see [Acz88] for details.

Now suppose that we introduce, following [SmT94], additional arity predicates F for every finite set $F \subseteq Fea$. The interpretation $F_{\mathcal{R}}$ of F consists of all feature trees t where the root of t has a label $L \in Lab$ and where F is (exactly) the set of all features departing from the root of t. Let Δ be the extended signature. Then $(\mathcal{R}^{\Delta}, End_{\mathcal{R}}^{\Delta}, X)$ is a strong SC-structure.

Let us now establish some useful formal properties of SC-structures.

Lemma 13. *Let $(\mathcal{A}^{\Sigma}, \mathcal{M}, X)$ be an SC-structure.*

1. $\mathcal{A}^{\Sigma} = SH_{\mathcal{M}}^{\mathcal{A}}(X)$ *and every mapping $X \to A$ has a* unique *extension to an endomorphism of $\mathcal{A}^{\Sigma}$ in $\mathcal{M}$.*
2. *For all finite sets $\{a_1, \ldots, a_n\} \subseteq A$ there exists a* unique minimal *finite subset Y of X such that $\{a_1, \ldots, a_n\} \subseteq SH_{\mathcal{M}}^{\mathcal{A}}(Y)$. This set will be called the* stabilizer $Stab_{\mathcal{M}}(a_1, \ldots, a_n)$ *of $\{a_1, \ldots, a_n\}$ with respect to $\mathcal{M}$.*

Using this notion of stabilizers, the validity of positive formulae in SC-structures can be characterized in an algebraic way. This characterization is essential for proving correctness of our combination method for constraint solvers over SC-structures. In the following lemma, letters of the form $\mathbf{u}$ and $\mathbf{v}$ ($\mathbf{e}$ and $\mathbf{x}$) denote sequences of variables (elements) of finite, non-fixed length.

Lemma 14. *Let $(\mathcal{A}^{\Sigma}, \mathcal{M}, X)$ be an SC-structure, and let*

$$\forall \mathbf{u}_1 \exists \mathbf{v}_1 \ldots \forall \mathbf{u}_k \exists \mathbf{v}_k \ \varphi(\mathbf{u}_1, \mathbf{v}_1, \ldots, \mathbf{u}_k, \mathbf{v}_k)$$

be a positive Σ-sentence. Then the following conditions are equivalent:

1. $\mathcal{A}^{\Sigma} \models \forall \mathbf{u}_1 \exists \mathbf{v}_1 \ldots \forall \mathbf{u}_k \exists \mathbf{v}_k \ \varphi(\mathbf{u}_1, \mathbf{v}_1, \ldots, \mathbf{u}_k, \mathbf{v}_k)$,
2. *there exist $\mathbf{x}_1 \in \mathbf{X}, \mathbf{e}_1 \in \mathbf{A}, \ldots, \mathbf{x}_k \in \mathbf{X}, \mathbf{e}_k \in \mathbf{A}$ such that*
   (a) $\mathcal{A}^{\Sigma} \models \varphi(\mathbf{x}_1, \mathbf{e}_1, \ldots, \mathbf{x}_k, \mathbf{e}_k)$,
   (b) *all $\mathcal{M}$-atoms in the sequences $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are distinct,*
   (c) *for all $j, 1 \leq j \leq k$, the components of $\mathbf{x}_j$ are not contained in $Stab_{\mathcal{M}}(\mathbf{e}_1) \cup \ldots \cup Stab_{\mathcal{M}}(\mathbf{e}_{j-1})$.*

The role of the second condition is perhaps not easy to grasp. Consider a prefix $\mathbf{x}_1, \mathbf{e}_1, \ldots, \mathbf{x}_{i-1}, \mathbf{e}_{i-1}, \mathbf{x}_i$ of the sequence in Condition 2. Parts (b) and (c) say that the atoms in $\mathbf{x}_i$ do not occur in the stabilizers of the elements $\mathbf{x}_1, \mathbf{e}_1, \ldots, \mathbf{x}_{i-1}, \mathbf{e}_{i-1}$ preceeding $\mathbf{x}_i$ in the linear order. In the proof, this fact is used to show that the elements in $\mathbf{x}_i$ may be mapped to arbitrary elements of A by surjective $\mathcal{M}$-endomorphisms that fix all the predecessors $\mathbf{x}_1, \mathbf{e}_1, \ldots, \mathbf{x}_{i-1}, \mathbf{e}_{i-1}$ at the same time.

In Section 5, where we describe how to construct amalgamated products of SC-structures, we will have to embed a given SC-structure $(\mathcal{A}^{\Sigma}, \mathcal{M}, X)$ in a larger SC-structure $(\mathcal{A}_{\infty}^{\Sigma}, \mathcal{M}_{\infty}, X_{\infty})$. Given $\mathcal{A}_{\infty}^{\Sigma}$, the amalgamated product will be obtained just by introducing additional functions and relations on this structure. The following, rather technical lemma collects all the conditions that are needed to establish later a collection of nice properties for the amalgam.

Lemma 15. *Let $(\mathcal{A}^{\Sigma}, \mathcal{M}, X)$ be an SC-structure. Then there exists an SC-structure $(\mathcal{A}^{\Sigma}_{\infty}, \mathcal{M}_{\infty}, X_{\infty})$ such that:*

(a0) $\mathcal{A}^{\Sigma}$ *and* $\mathcal{A}^{\Sigma}_{\infty}$ *are isomorphic.*
(a1) $\mathcal{A}^{\Sigma} = SH^{\mathcal{A}_{\infty}}_{\mathcal{M}_{\infty}}(X)$, $X \subset X_{\infty}$, *and* $X_{\infty} \setminus X$ *is infinite.*
(a2) $(\mathcal{A}^{\Sigma}_{\infty}, \mathcal{M}_{\infty}, X_{\infty})$ *is strong iff* $(\mathcal{A}^{\Sigma}, \mathcal{M}, X)$ *is strong.*
(a3) If $(\mathcal{A}^{\Sigma}, \mathcal{M}, X)$ *is a strong SC-structure, then every mapping* $X \to A_{\infty}$ *has a unique extension to a homomorphism* $h^{\Sigma}_{A-A_{\infty}} : \mathcal{A}^{\Sigma} \to \mathcal{A}^{\Sigma}_{\infty}$.
(a4) If $(\mathcal{A}^{\Sigma}, \mathcal{M}, X)$ *is a strong SC-structure, and if* $X \subseteq X' \subseteq X_{\infty}$, *then every bijection* $g_0 : X \to X'$ *has a unique extension to an isomorphism between* $SH^{\mathcal{A}_{\infty}}_{\mathcal{M}_{\infty}}(X)$ *and* $SH^{\mathcal{A}_{\infty}}_{\mathcal{M}_{\infty}}(X')$.

For the case of a term algebra modulo an equational theory, the statement of the lemma trivially holds. In fact, if V_{∞} is any countable superset of the countably infinite set V, then $\mathcal{T}(\Sigma_F, V)/{=_E}$ is isomorphic to $\mathcal{T}(\Sigma_F, V_{\infty})/{=_E}$. In the case of SC-structures, the proof is much more involved.

5 Amalgamation of Simply Combinable Structures

We describe an explicit construction that may be used to close any amalgamation base where the two components are SC-structures over disjoint signatures. If both components are strong SC-structures, then this construction yields the free amalgamated product of these structures. In the general case, the resulting structure also seems to play a unique role, but a precise characterization of this intuition has not yet been obtained. The construction is almost identical to the amalgamation construction given in [BaS94a] for the case of free structures. There is just one essential difference. In [BaS94a], substructures that are generated by increasing sets of free generators are used in each step of the construction. Here, in the case of SC-structures, stable hulls (as defined in Definition 8) of increasing sets of atoms must be used instead.

Let $(\mathcal{A}^{\Sigma}, \mathcal{M}, X)$ and $(\mathcal{B}^{\Delta}, \mathcal{N}, X)$ be two SC-structures over disjoint signatures Σ and Δ. We consider the amalgamation base $(X, \mathcal{A}^{\Sigma}, \mathcal{B}^{\Delta})$, where the common part is just the set of atoms X. Thus, the embedding "homomorphisms" $h_{X-A} : X \to A^{\Sigma}$ and $h_{X-B} : X \to B^{\Delta}$ are given by Id_X, i.e., the identity mapping on X. In order to close this amalgamation base, we shall first embed $\mathcal{A}^{\Sigma}$ and $\mathcal{B}^{\Delta}$ into isomorphic superstructures. Let $(\mathcal{A}^{\Sigma}_{\infty}, \mathcal{M}_{\infty}, X_{\infty})$ be an SC-superstructure of $(\mathcal{A}^{\Sigma}, \mathcal{M}, X)$ satisfying conditions (a0)–(a4) of Lemma 15. Analogously, there exists an SC-superstructure $(\mathcal{B}^{\Delta}_{\infty}, \mathcal{N}_{\infty}, Y_{\infty})$ of $(\mathcal{B}^{\Delta}, \mathcal{N}, X)$ such that the corresponding properties (b0)–(b4) hold.

Starting from $\mathcal{A}^{\Sigma}_0 := \mathcal{A}^{\Sigma}$ and $\mathcal{B}^{\Delta}_0 := \mathcal{B}^{\Delta}$, we shall make a zig-zag construction that defines an ascending tower of Σ-structures $\mathcal{A}^{\Sigma}_n$, and similarly an ascending tower of Δ-structures $\mathcal{B}^{\Delta}_n$. These structures are connected by bijective mappings h_n and g_n. The combined structure is obtained as the limit structure, which obtains its functional and relational structure from both towers by means of the limits of the mappings h_n and g_n. Let $X_0 := Y_0 := X$.

$n = 0$: Consider $\mathcal{A}_0^{\Sigma} = \mathcal{A}^{\Sigma} = SH_{\mathcal{M}_\infty}^{\mathcal{A}_\infty}(X_0)$. We interpret the "new" elements in $A_0 \setminus X_0$ as atoms in $\mathcal{B}_\infty^{\Delta}$. For this purpose, select a subset $Y_1 \subseteq Y_\infty$ such that $Y_1 \cap Y_0 = \emptyset$, $|Y_1| = |A_0 \setminus X_0|$, and the remaining complement $Y_\infty \setminus (Y_0 \cup Y_1)$ is countably infinite. Choose any bijection $h_0 : Y_0 \cup Y_1 \to A_0$ where $h_0|_{Y_0} = Id_{Y_0}$.

Consider $\mathcal{B}_0^{\Delta} = \mathcal{B}^{\Delta} = SH_{\mathcal{N}_\infty}^{\mathcal{B}_\infty}(Y_0)$. As for $\mathcal{A}_0$, we interpret the "new" elements in $B_0 \setminus Y_0$ as atoms in $\mathcal{A}_\infty^{\Sigma}$. Select a subset $X_1 \subseteq X_\infty$ such that $X_1 \cap X_0 = \emptyset$, $|X_1| = |B_0 \setminus Y_0|$ and the remaining complement $X_\infty \setminus (X_0 \cup X_1)$ is countably infinite. Choose any bijection $g_0 : X_0 \cup X_1 \to B_0$ where $g_0|_{X_0} = Id_{X_0}$.

$n \to n+1$: Suppose that the structures $\mathcal{A}_n^{\Sigma} = SH_{\mathcal{M}_\infty}^{\mathcal{A}_\infty}(\bigcup_{i=0}^{n} X_i)$ and $\mathcal{B}_n^{\Delta} = SH_{\mathcal{N}_\infty}^{\mathcal{B}_\infty}(\bigcup_{i=0}^{n} Y_i)$ and the atom sets $X_{n+1} \subset (X_\infty \setminus \bigcup_{i=0}^{n} X_i)$ and $Y_{n+1} \subset (Y_\infty \setminus \bigcup_{i=0}^{n} Y_i)$ are already defined. We assume that the complements $X_\infty \setminus \bigcup_{i=0}^{n+1} X_i$ and $Y_\infty \setminus \bigcup_{i=0}^{n+1} Y_i$ are infinite. In addition, we assume that bijections $h_n : B_{n-1} \cup Y_n \cup Y_{n+1} \to A_n$ and $g_n : A_{n-1} \cup X_n \cup X_{n+1} \to B_n$ are defined such that

$$(*)\ g_n(h_n(b)) = b \text{ for } b \in B_{n-1} \cup Y_n \text{ and } h_n(g_n(a)) = a \text{ for } a \in A_{n-1} \cup X_n,$$
$$(**)\ h_n(Y_{n+1}) = A_n \setminus (A_{n-1} \cup X_n) \quad \text{and } g_n(X_{n+1}) = B_n \setminus (B_{n-1} \cup Y_n).$$

We define $\mathcal{A}_{n+1}^{\Sigma} := SH_{\mathcal{M}_\infty}^{\mathcal{A}_\infty}(\bigcup_{i=0}^{n+1} X_i)$ and $\mathcal{B}_{n+1}^{\Delta} = SH_{\mathcal{N}_\infty}^{\mathcal{B}_\infty}(\bigcup_{i=0}^{n+1} Y_i)$ and select subsets $Y_{n+2} \subseteq Y_\infty$ and $X_{n+2} \subseteq X_\infty$ such that $Y_{n+2} \cap \bigcup_{i=0}^{n+1} Y_i = \emptyset = X_{n+2} \cap \bigcup_{i=0}^{n+1} X_i$. In addition, the cardinalities must satisfy $|Y_{n+2}| = |A_{n+1} \setminus (A_n \cup X_{n+1})|$ and $|X_{n+2}| = |B_{n+1} \setminus (B_n \cup Y_{n+1})|$, and the remaining complements $Y_\infty \setminus \bigcup_{i=0}^{n+2} Y_i$ and $X_\infty \setminus \bigcup_{i=0}^{n+2} X_i$ must be countably infinite. Let

$$\upsilon_{n+1} : Y_{n+2} \to A_{n+1} \setminus (A_n \cup X_{n+1}) \quad \text{and} \quad \xi_{n+1} : X_{n+2} \to B_{n+1} \setminus (B_n \cup Y_{n+1})$$

be arbitrary bijections. We define $h_{n+1} := \upsilon_{n+1} \cup g_n^{-1} \cup h_n$ and $g_{n+1} := \xi_{n+1} \cup h_n^{-1} \cup g_n$.

Without loss of generality we may assume (for notational convenience) that the construction eventually covers all atoms in X_∞ and Y_∞; in other words, we assume that $\bigcup_{i=0}^{\infty} X_i = X_\infty$ and $\bigcup_{i=0}^{\infty} Y_i = Y_\infty$, and thus $\bigcup_{i=0}^{\infty} A_i = A_\infty$ and $\bigcup_{i=0}^{\infty} B_i = B_\infty$. We define the limit mappings $h_\infty := \bigcup_{i=0}^{\infty} h_i : B_\infty \to A_\infty$ and $g_\infty := \bigcup_{i=0}^{\infty} g_i : A_\infty \to B_\infty$. It is easy to see that h_∞ and g_∞ are bijections that are inverse to each other. They may be used to carry the Δ-structure of $\mathcal{B}_\infty^{\Delta}$ to $\mathcal{A}_\infty^{\Sigma}$, and to carry the Σ-structure of $\mathcal{A}_\infty^{\Sigma}$ to $\mathcal{B}_\infty^{\Delta}$: Let f (f') be an n-ary function symbol of Δ (Σ), let p (p') be an n-ary predicate symbol of Δ (Σ), and let $a_1, \ldots, a_n \in A_\infty$ ($b_1, \ldots, b_n \in B_\infty$). We define

$$\begin{aligned}
f_{\mathcal{A}_\infty}(a_1, \ldots, a_n) &:= h_\infty(f_{\mathcal{B}_\infty}(g_\infty(a_1), \ldots, g_\infty(a_n))),\\
f'_{\mathcal{B}_\infty}(b_1, \ldots, b_n) &:= g_\infty(f'_{\mathcal{A}_\infty}(h_\infty(b_1), \ldots, h_\infty(b_n))),\\
p_{\mathcal{A}_\infty}[a_1, \ldots, a_n] &:\Longleftrightarrow p_{\mathcal{B}_\infty}[g_\infty(a_1), \ldots, g_\infty(a_n)],\\
p'_{\mathcal{B}_\infty}[b_1, \ldots, b_n] &:\Longleftrightarrow p'_{\mathcal{A}_\infty}[h_\infty(b_1), \ldots, h_\infty(b_n)].
\end{aligned}$$

With this definition, the mappings h_∞ and g_∞ are inverse isomorphisms between the $(\Sigma \cup \Delta)$-structures $\mathcal{A}_\infty^{\Sigma\cup\Delta}$ and $\mathcal{B}_\infty^{\Sigma\cup\Delta}$. We take $\mathcal{A}_\infty^{\Sigma\cup\Delta}$ as the result of the construction.

Lemma 16. *$\mathcal{A}_\infty^{\Sigma\cup\Delta}$ closes the amalgamation base $(X, \mathcal{A}^{\Sigma}, \mathcal{B}^{\Delta})$.*

Our assumption (a_0) on $\mathcal{A}_\infty^\Sigma$ shows that $\mathcal{A}^\Sigma$ and $\mathcal{A}_\infty^\Sigma$ have the same first order theory. Similarly (b_0) shows that $\mathcal{B}^\Delta$ and $\mathcal{B}_\infty^\Delta$ or $\mathcal{A}_\infty^\Delta$ have the same first order theory. Thus, from a logical point of view the relationship between the components $\mathcal{A}^\Sigma$ and $\mathcal{B}^\Delta$ and the amalgam $\mathcal{A}_\infty^{\Sigma\cup\Delta}$ is optimal. In order to obtain a better algebraic characterization of what the above construction generates, we restrict our attention to strong SC-structures. First, we must define a class of admissible structures. To this purpose we use the algebraic condition of Proposition 2:

Definition 17. For strong SC-structures $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ and $(\mathcal{B}^\Delta, \mathcal{N}, X)$, the class of admissible structures, $Adm(\mathcal{A}^\Sigma, \mathcal{B}^\Delta)$, consists of all structures $\mathcal{C}^{\Sigma\cup\Delta}$ such that for every mapping $g_{X-C} : X \to C$ there exist unique homomorphisms $g_{A-C}^\Sigma : \mathcal{A}^\Sigma \to \mathcal{C}^\Sigma$ and $g_{B-C}^\Delta : \mathcal{B}^\Delta \to \mathcal{C}^\Delta$ extending g_{X-C}.

We may now formulate our central result concerning amalgamation of strong SC-structures. In the proof, the conditions $(a_1)-(a_4)$ and $(b_1)-(b_4)$ that have been imposed on $\mathcal{A}_\infty^\Sigma$ and $\mathcal{B}_\infty^\Delta$ at the beginning of the amalgamation construction become relevant.

Theorem 18. *If $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ and $(\mathcal{B}^\Delta, \mathcal{N}, X)$ are strong SC-structures over disjoint signatures, then $\mathcal{A}_\infty^{\Sigma\cup\Delta}$ is the free amalgamated product of $\mathcal{A}^\Sigma$ and $\mathcal{B}^\Delta$ over X with respect to the class $\mathrm{Adm}(\mathcal{A}^\Sigma, \mathcal{B}^\Delta)$ of admissible structures defined above.*

For strong SC-structures, the amalgamation construction can be applied iteratedly because the obtained structure is again a strong SC-structure:

Theorem 19. *The free amalgamated product of two strong SC-structures with common atom set X is a strong SC-structure with atom set X.*

The following theorem is needed to prove correctness of our method for deciding positive constraints over the free amalgamated product of two strong SC-structures with disjoint signatures.

Theorem 20. *Free amalgamation of strong SC-structures with disjoint signatures over the same atom set is associative.*

6 Combining Constraint Solvers for SC-Structures

Let $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ and $(\mathcal{B}^\Delta, \mathcal{N}, X)$ be two SC-structures over disjoint signatures Σ and Δ; let $\mathcal{A}^\Sigma \otimes B^\Delta \simeq \mathcal{A}_\infty^{\Sigma\cup\Delta}$ denote the result of the amalgamation construction described in the previous section.

Lemma 21. *There exists a decomposition algorithm that decomposes a positive existential $(\Sigma\cup\Delta)$-sentence φ_0 into a finite set of output pairs (α, β), where α is a positive Σ-sentence, and β is a positive Δ-sentence, such that $\mathcal{A}^\Sigma \otimes \mathcal{B}^\Delta \models \varphi_0$ iff $\mathcal{A}^\Sigma \models \alpha$ and $\mathcal{B}^\Delta \models \beta$ for some output pair (α, β).*

A brief description of the algorithm is given in the Appendix. A detailled description of all steps can be found in [BaS94a], where the same algorithm has been used in the restricted context of constraint solvers for free structures.

Theorem 22. *The existential positive theory of* $\mathcal{A}^\Sigma \otimes \mathcal{B}^\Delta$ *is decidable, provided that the positive theories of* $\mathcal{A}^\Sigma$ *and of* $\mathcal{B}^\Delta$ *are decidable.*

Recall that, for strong SC-structures $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ and $(\mathcal{B}^\Delta, \mathcal{N}, X)$, the structure $\mathcal{A}^\Sigma \otimes \mathcal{B}^\Delta$ is the free amalgamated product $\mathcal{A}^\Sigma \odot \mathcal{B}^\Delta$of $\mathcal{A}^\Sigma$ and $\mathcal{B}^\Delta$ over X with respect to $Adm(\mathcal{A}^\Sigma, \mathcal{B}^\Delta)$. In this case, our combination method is not restricted to *existential* positive sentences. The main idea is to transform positive sentences (with arbitrary quantifier prefix) into existential positive sentences by Skolemizing the universally quantified variables. In principle, the decomposition algorithm for positive sentences is now applied twice to decompose the input sentence into three positive sentences α, β, ρ, whose validity must respectively be decided in $\mathcal{A}^\Sigma$, $\mathcal{B}^\Delta$, and the absolutely free term algebra over the Skolem functions (here Theorem 20 becomes relevant).

Theorem 23. *If* $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ *and* $(\mathcal{B}^\Delta, \mathcal{N}, X)$ *are strong SC-structures then the (full) positive theory of* $\mathcal{A}^\Sigma \odot \mathcal{B}^\Delta$ *is decidable, provided that the positive theories of* $\mathcal{A}^\Sigma$ *and of* $\mathcal{B}^\Delta$ *are decidable.*

In connection with the Theorems 19 and 20, this provides the basis for constraint solving in the combination of any finite number of strong SC-structures.

Theorems 22 and 23 show that the prerequisite for combining constraint solvers with the help of our decomposition algorithms is that validity of arbitrary positive sentences is decidable in both components. If we leave the realm of free structures, not many results are known that show that the positive theory of a particular SC-structure is decidable. One example is the algebra of rational trees: its full first order theory—like the theory of the algebra of finite trees—is known to be decidable [Mah88].[7] In general, the problem of deciding validity of existential positive sentences and the problem of deciding validity of arbitrary positive sentences in a given structure can be quite different. For the case of SC-structures, however, the following variant of Lemma 14 shows that the difference is not drastic.

Lemma 24. *Let* $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ *be an SC-structure, let*

$$\forall \mathbf{u}_1 \exists \mathbf{v}_1 \ldots \forall \mathbf{u}_k \exists \mathbf{v}_k \ \varphi(\mathbf{u}_1, \mathbf{v}_1, \ldots, \mathbf{u}_k, \mathbf{v}_k)$$

be a positive Σ*-sentence, and let, for each* $i, 1 \leq i \leq k$, $\mathbf{x}_i$ *be an arbitrary (but fixed) sequence of length* $|\mathbf{u}_i|$ *of distinct atoms such that distinct sequences* $\mathbf{x}_i$ *and* $\mathbf{x}_j$ *do not have common elements. Let* $X_{1,i}$ *denote the set of all atoms occurring in the sequences* $\mathbf{x}_1, \ldots, \mathbf{x}_i$ $(i = 1, \ldots, k)$*. Then the following conditions are equivalent:*

1. $\mathcal{A}^\Sigma \models \forall \mathbf{u}_1 \exists \mathbf{v}_1 \ldots \forall \mathbf{u}_k \exists \mathbf{v}_k \ \varphi(\mathbf{u}_1, \mathbf{v}_1, \ldots, \mathbf{u}_k, \mathbf{v}_k)$,

[7] Maher considers ground tree algebras, but over possibly infinite signatures. Therefore his result can be lifted to the non-ground case by treating variables as constants.

2. there exist $\mathbf{e}_1 \in SH^{\mathcal{A}}_{\mathcal{M}}(X_{1,1}), \ldots, \mathbf{e}_k \in SH^{\mathcal{A}}_{\mathcal{M}}(X_{1,k})$ *such that* $\mathcal{A}^{\Sigma} \models \varphi(\mathbf{x}_1, \mathbf{e}_1, \ldots, \mathbf{x}_k, \mathbf{e}_k)$.

Looking at the second condition of the lemma, one sees that a positive sentence can be reduced to an *existential* positive sentence where the universally quantified variables are replaced by atoms (i.e., free constants), and additional restrictions are imposed on the values of the existentially quantified variables. For this reason, it is often not hard to extend decision procedures for the existential positive theory of an SC-structure to a decision procedure for the full positive theory. This way of proceeding can, for example, be used to prove that the positive theories of the four domains of nested, heriditarily finite wellfounded or non-wellfounded sets or lists, as introduced in Example 12, are decidable.

Corollary 25. *Simultaneous free amalgamated products have a decidable positive theory if the components are finite or rational tree algebras, or nested, heriditarily finite wellfounded or non-wellfounded sets or lists, and if the signatures of the components are disjoint.*

7 Conclusion

This paper should be seen as a first step to provide an abstract framework for the combination of constraint languages and constraint solvers. We have introduced the notion "admissible amalgamated product" in order to capture—in an abstract algebraic setting—our intuition of what a combined solution structure should satisfy. It was shown that in certain cases there exists a canonical structure—called the free amalgamated product—that yields a most general admissible closure of a given amalgamation base.

We have introduced a class of structures—called SC-structures—that are equipped with structural properties that guarantee (1) that a canonical amalgamation construction can be applied to SC-structures over disjoint signatures, and (2) that validity of positive existential formulae in the amalgamated structure obtained by this construction can be reduced to validity of positive formulae in the component structures. For the subclass of strong SC-structures we have obtained stronger results. Interestingly, a very similar class of structures has independently been introduced in [ScS88, Wil91] in order to characterize a maximal class of algebras where equation (and constraint) solving essentially behaves like unification.[8]

It is interesting to compare the concrete combined solution domains that can be found in the literature with the combined domains obtained by our amalgamation construction. It turns out that there can be differences if the elements of the components have a tree-like structure that allows for infinite paths (as in the examples of non-wellfounded lists/sets and rational trees). In these cases, frequently a combined solution structure is chosen where an infinite number of

[8] The notion of an SC-structure can be considered as a sort-free version of the concepts that have been discussed in [ScS88, Wil91].

"signature changes" may occur when following an infinite path in an element of the combined domain ([Col90, Rou88]). In contrast, our amalgamation construction yields a combined structure where elements allow for a finite number of signature changes only. This indicates that the free amalgamated product, even if it exists, is not necessarily the only interesting combined domain. It remains to be seen which additional natural ways to combine structures exist, and how different ways of combining structures are formally related.

It should be noted that for most of the results presented in the paper the presence of countably many atoms ("variables") in the structures to be combined is an essential precondition. On the other hand, many constraint-based approaches consider ground structures as solution domains. In most cases, however, a corresponding non-ground structure containing the necessary atoms exists. Thus, our combination method can be applied to these non-ground variants. Of course, the combined structure obtained in this way is again non-ground. However, in the context of constraint solving this distinction is rather irrelevant: typically, "constraints" are *existential* positive formulae, and for existential positive formulae, validity in the non-ground combined structure is equivalent to validity in the ground variant of the combined structure.[9] This observation has the following interesting consequence. Even in cases where the (full) positive theory of a ground component structure is undecidable, our combination methods can be applied to show decidability of the existential positive theory even for the *ground* combined structure, provided that the (full) positive theories of the non-ground component structures are decidable. Our remark following Lemma 24 shows that decidability of the full positive theory of such a non-ground structure can sometimes be obtained by an easy modification of the decision method for the existential positive case. Free semigroups are an example for this situation: the positive theory of a free semigroup with a finite number $n \geq 2$ of generators is undecidable, whereas the positive theory of the countably generated free semigroup (which corresponds to our non-ground case) is decidable [VaR83].

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Appendix: The Decomposition Algorithm Let φ_0 be a positive existential $(\Sigma \cup \Delta)$-sentence, the *input*. We may assume that φ_0 has the form $\exists \mathbf{u}_0\ \gamma_0$, where γ_0 is a conjunction of atomic formulae.

Step 1. An equivalent positive existential $(\Sigma \cup \Delta)$-sentence φ_1 is generated where all atomic subformulae are pure, i.e., they are built over one signature (Σ or Δ) only.

Step 2. All equations $u = v$ between variables are removed after replacing every occurrence of u in φ_1 by v. Let φ_2 be the new sentence obtained this way. The matrix of φ_2 can be written as a conjunction $\gamma_{2,\Sigma} \wedge \gamma_{2,\Delta}$, where $\gamma_{2,\Sigma}$ is the conjunction of all atomic Σ-subformulae, and $\gamma_{2,\Delta}$ is the conjunction of all atomic Δ-subformulae. There are three different types of variables occurring in φ_2: shared variables occur both in $\gamma_{2,\Sigma}$ and in $\gamma_{2,\Delta}$; Σ-variables (Δ-variables) occur only in $\gamma_{2,\Sigma}$ (in $\gamma_{2,\Delta}$). Let $\mathbf{u}_{2,\Sigma}$ ($\mathbf{u}_{2,\Delta}$) be the tuple of all Σ-variables (Δ-variables), let $\mathbf{u}_2$ be the tuple of all shared variables. Obviously, φ_2 is equivalent to the sentence $\exists \mathbf{u}_2\ (\exists \mathbf{u}_{2,\Sigma}\ \gamma_{2,\Sigma} \wedge \exists \mathbf{u}_{2,\Delta}\ \gamma_{2,\Delta})$.

Step 3 (non-deterministic). We choose a partition of the set of *shared* variables. For each class of the partition, a representative is selected, and all variables of the class are replaced by the representative. Quantifiers for replaced variables are removed. Let $\exists \mathbf{u}_3\ (\exists \mathbf{u}_{2,\Sigma}\ \gamma_{3,\Sigma} \wedge \exists \mathbf{u}_{2,\Delta}\ \gamma_{3,\Delta})$ denote a sentence obtained by Step 3.

Step 4 (non-deterministic). We choose a label Σ or Δ for each component of $\mathbf{u}_3$, and a linear ordering $<$ on the set of these variables.

Step 5. The sentence $\exists \mathbf{u}_3 (\exists \mathbf{u}_{2,\Sigma}\ \gamma_{3,\Sigma} \wedge \exists \mathbf{u}_{2,\Delta}\ \gamma_{3,\Delta})$ is split into two sentences

$$\alpha = \forall \mathbf{v}_1 \exists \mathbf{w}_1 \ldots \forall \mathbf{v}_k \exists \mathbf{w}_k \exists \mathbf{u}_{2,\Sigma}\ \gamma_{3,\Sigma}, \quad \text{and} \quad \beta = \exists \mathbf{v}_1 \forall \mathbf{w}_1 \ldots \exists \mathbf{v}_k \forall \mathbf{w}_k \exists \mathbf{u}_{2,\Delta}\ \gamma_{3,\Delta}.$$

Here $\mathbf{v}_1 \mathbf{w}_1 \ldots \mathbf{v}_k \mathbf{w}_k$ is the unique re-ordering of $\mathbf{u}_3$ along $<$. The variables $\mathbf{v}_i$ ($\mathbf{w}_i$) are the variables with label Δ (label Σ). The *output sentences* α and β are (not necessarily existential) positive formulae.