Context-Free Languages

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Slides based on RPI CSCI 2400

Thanks to Costas Busch

Take-away

Definition context-free grammar

Definition context-free language

Derivation, sentential form, sentence

Derivation trees

Ambiguity

Context-free grammars for natural language

Terminology

For our purposes in this class:

Context free grammar

=

Constituency grammar

=

Phrase structure grammar

Grammars

Grammars express languages

Example: the English language

$$\langle sentence \rangle \rightarrow \langle noun\_phrase \rangle \ \langle predicate \rangle$$

$$\langle noun\_phrase \rangle \rightarrow \langle article \rangle \ \langle noun \rangle$$

$$\langle predicate \rangle \rightarrow \langle verb \rangle$$

$$\langle article \rangle \to a$$

$$\langle article \rangle \to the$$

$$\langle noun \rangle \to cat$$

$$\langle noun \rangle \to dog$$

$$\langle verb \rangle \to runs$$

$$\langle verb \rangle \to walks$$

A derivation of "the dog walks":

$$\begin{aligned}
\langle sentence \rangle &\Rightarrow \langle noun_phrase \rangle \ \langle predicate \rangle \\
&\Rightarrow \langle noun_phrase \rangle \ \langle verb \rangle \\
&\Rightarrow \langle article \rangle \ \langle noun \rangle \ \langle verb \rangle \\
&\Rightarrow the \ \ \langle noun \rangle \ \langle verb \rangle \\
&\Rightarrow the \ \ dog \ \ \langle verb \rangle \\
&\Rightarrow the \ \ dog \ \ walks
\end{aligned}$$

A derivation of "a cat runs":

$$\begin{aligned}\langle sentence\rangle &\Rightarrow \langle noun\_phrase\rangle\ \langle predicate\rangle \\ &\Rightarrow \langle noun\_phrase\rangle\ \langle verb\rangle \\ &\Rightarrow \langle article\rangle\ \langle noun\rangle\ \langle verb\rangle \\ &\Rightarrow a\ \ \langle noun\rangle\ \langle verb\rangle \\ &\Rightarrow a\ \ cat\ \ \langle verb\rangle \\ &\Rightarrow a\ \ cat\ \ runs\end{aligned}$$

Language of the grammar:

L = { "a cat runs",
"a cat walks",
"the cat runs",
"the cat walks",
"a dog runs",
"a dog walks",
"the dog runs",
"the dog walks" }

Notation

Production Rules

$$\langle noun \rangle \rightarrow cat$$

$$\langle noun \rangle \rightarrow dog$$

Variable

Terminal

Another Example

Grammar:

$$S \rightarrow aSb$$

$$S \rightarrow \lambda$$

Derivation of sentence ab:

$$S \Rightarrow aSb \Rightarrow ab$$

$$S \rightarrow aSb \qquad\qquad S \rightarrow \lambda$$

Language?

Grammar: $S \rightarrow aSb$

$S \rightarrow \lambda$

Derivation of sentence $aabb$:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aabb$$

$S \rightarrow aSb$ $\qquad$ $S \rightarrow \lambda$

Other derivations:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb$$

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb$$
$$\Rightarrow aaaaSbbbb \Rightarrow aaaabbbb$$

Language of the grammar

$$S \to aSb$$

$$S \to \lambda$$

$$L = \{a^n b^n : n \geq 0\}$$

More Notation

Grammar $$G = (V, T, S, P)$$

$V:$ Set of variables

$T:$ Set of terminal symbols

$S:$ Start variable

$P:$ Set of Production rules

Example

Grammar G :

$$S \rightarrow aSb$$

$$S \rightarrow \lambda$$

$$G = (V, T, S, P)$$

$$V = \{S\} \qquad T = \{a, b\}$$

$$P = \{S \rightarrow aSb,\ S \rightarrow \lambda\}$$

More Notation

Sentential Form:

A sentence that contains variables and terminals

Example:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb$$

Sentential Forms — sentence

We write: $S \overset{*}{\Rightarrow} aaabbb$

Instead of:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb$$

In general we write: $w_1 \overset{*}{\Rightarrow} w_n$

If: $w_1 \Rightarrow w_2 \Rightarrow w_3 \Rightarrow \cdots \Rightarrow w_n$

By default: $w \overset{*}{\Longrightarrow} w$

Example

Grammar

$$S \to aSb$$

$$S \to \lambda$$

Derivations

$$S \overset{*}{\Rightarrow} \lambda$$

$$S \overset{*}{\Rightarrow} ab$$

$$S \overset{*}{\Rightarrow} aabb$$

$$S \overset{*}{\Rightarrow} aaabbb$$

Example

Grammar

$$S \rightarrow aSb$$

$$S \rightarrow \lambda$$

Derivations

$$S \overset{*}{\Rightarrow} aaSbb$$

$$aaSbb \overset{*}{\Rightarrow} aaaaaSbbbbb$$

Another Grammar Example

Grammar G:

$$S \to Ab$$

$$A \to aAb$$

$$A \to \lambda$$

Language?

Grammar G:
$$S \rightarrow Ab$$
$$A \rightarrow aAb$$
$$A \rightarrow \lambda$$

Derivations:

$$S \rightarrow Ab \rightarrow b$$
$$S \rightarrow Ab \rightarrow aAbb \rightarrow abb$$
$$S \rightarrow Ab \rightarrow aAbb \rightarrow aaAbbb \rightarrow aabbb$$

More Derivations

$$S \Rightarrow Ab \Rightarrow aAbb \Rightarrow aaAbbb \Rightarrow aaaAbbbb$$

$$\Rightarrow aaaaAbbbbb \Rightarrow aaaabbbbb$$

$$S \overset{*}{\Rightarrow} aaaabbbbb$$

$$S \overset{*}{\Rightarrow} aaaaaabbbbbbb$$

$$S \overset{*}{\Rightarrow} a^n b^n b$$

Language of a Grammar

For a grammar G
with start variable S :

$$L(G) = \{w: \quad S \overset{*}{\Rightarrow} w\}$$

↑
String of terminals

Example

For grammar G : $S \to Ab$

$$A \to aAb$$

$$A \to \lambda$$

$$L(G) = \{a^n b^n b : \quad n \geq 0\}$$

Since: $S \overset{*}{\Rightarrow} a^n b^n b$

A Convenient Notation

$$A \to aAb$$
$$A \to \lambda$$

$$\Longrightarrow \quad A \to aAb \mid \lambda$$

$$\langle article \rangle \to a$$
$$\langle article \rangle \to the$$

$$\Longrightarrow \quad \langle article \rangle \to a \mid the$$

Revisit first grammar

A context-free grammar G:

$$S \rightarrow aSb$$

$$S \rightarrow \lambda$$

A derivation:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aabb$$

A context-free grammar G:

$$S \rightarrow aSb$$

$$S \rightarrow \lambda$$

Another derivation:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb$$

$$S \rightarrow aSb$$

$$S \rightarrow \lambda$$

$$L(G) = \{a^n b^n : n \geq 0\}$$

Describes parentheses: (((( ))))

Example

A context-free grammar G:

$$S \to aSa$$

$$S \to bSb$$

$$S \to \lambda$$

Language?

A context-free grammar G:

$$S \rightarrow aSa$$

$$S \rightarrow bSb$$

$$S \rightarrow \lambda$$

Another derivation:

$$S \Rightarrow aSa \Rightarrow abSba \Rightarrow abaSaba \Rightarrow abaaba$$

$$S \to aSa$$

$$S \to bSb$$

$$S \to \lambda$$

$$L(G) = \{ww^R : \; w \in \{a,b\}^*\}$$

Example

A context-free grammar G:

$$S \rightarrow aSb$$

$$S \rightarrow SS$$

$$S \rightarrow \lambda$$

Language?

A context-free grammar G:

$$S \rightarrow aSb$$

$$S \rightarrow SS$$

$$S \rightarrow \lambda$$

Two derivations:

$$S \Rightarrow SS \Rightarrow aSbS \Rightarrow abS \Rightarrow abaSb \Rightarrow abab$$

$$S \Rightarrow SS \Rightarrow aSbS \Rightarrow abS \Rightarrow ab$$

$$S \to aSb$$

$$S \to SS$$

$$S \to \lambda$$

$$L(G) = \{w \; : n_a(w) = n_b(w),$$

$$\text{and } n_a(v) \geq n_b(v)$$

$$\text{in any prefix } v\}$$

Interpretation?

$$S \to aSb$$

$$S \to SS$$

$$S \to \lambda$$

$$L(G) = \{w \ : n_a(w) = n_b(w),$$

$$\text{and } n_a(v) \geq n_b(v)$$

$$\text{in any prefix } v\}$$

Describes matched parentheses: () ((( ))) (( ))

Definition: Context-Free Grammars

Grammar $G = (V, T, S, P)$

Variables — Terminal symbols — Start variable

Productions of the form:

$$A \to x$$

Variable — String of variables and terminals

$$G = (V, T, S, P)$$

$$L(G) = \{w : \quad S \overset{*}{\Rightarrow} w, \quad w \in T^*\}$$

Definition: Context-Free Languages

A language L is context-free

if and only if

there is a context-free grammar G
with $L = L(G)$

Derivation Order

1. $S \to AB$

2. $A \to aaA$

3. $A \to \lambda$

4. $B \to Bb$

5. $B \to \lambda$

Leftmost derivation:

$$S \overset{1}{\Rightarrow} AB \overset{2}{\Rightarrow} aaAB \overset{3}{\Rightarrow} aaB \overset{4}{\Rightarrow} aaBb \overset{5}{\Rightarrow} aab$$

Rightmost derivation:

$$S \overset{1}{\Rightarrow} AB \overset{4}{\Rightarrow} ABb \overset{5}{\Rightarrow} Ab \overset{2}{\Rightarrow} aaAb \overset{3}{\Rightarrow} aab$$

Language?

$$S \to aAB$$

$$A \to bBb$$

$$B \to A \mid \lambda$$

Leftmost derivation:

$$S \Rightarrow aAB \Rightarrow abBbB \Rightarrow abAbB \Rightarrow abbBbbB$$

$$\Rightarrow abbbbB \Rightarrow abbbb$$

Rightmost derivation:

$$S \Rightarrow aAB \Rightarrow aA \Rightarrow abBb \Rightarrow abAb$$

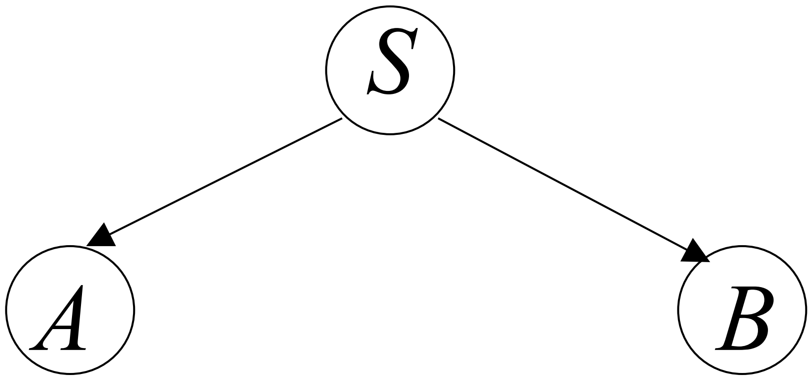
$$\Rightarrow abbBbb \Rightarrow abbbb$$

Language?

Derivation Trees

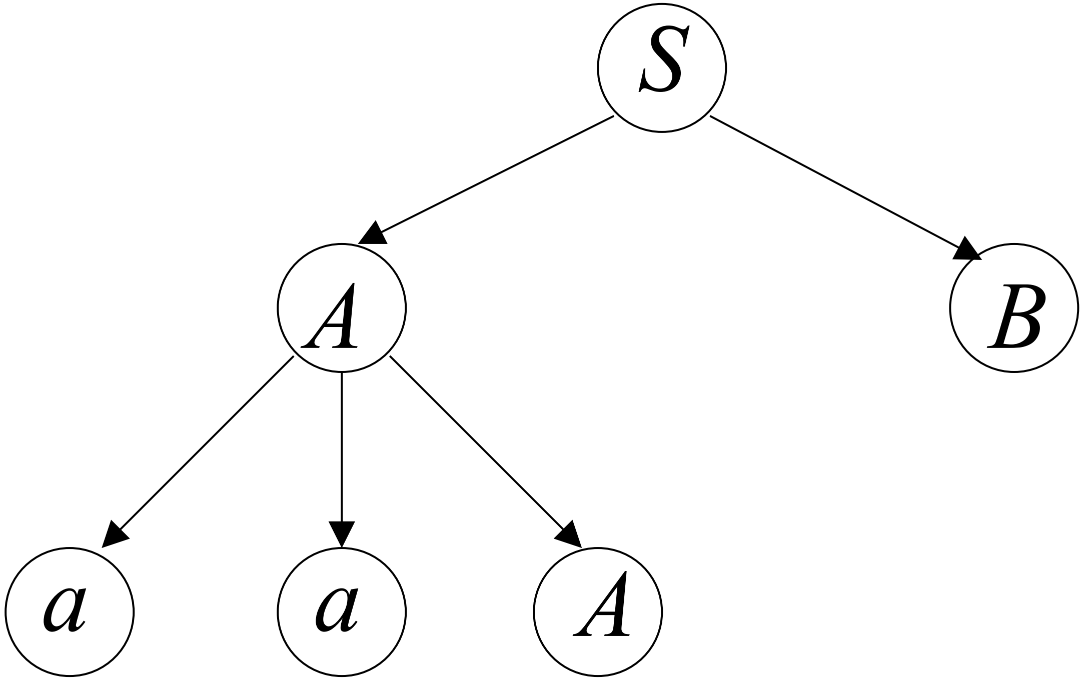
$S \to AB \qquad A \to aaA \mid \lambda \qquad B \to Bb \mid \lambda$

$$S \Rightarrow AB$$



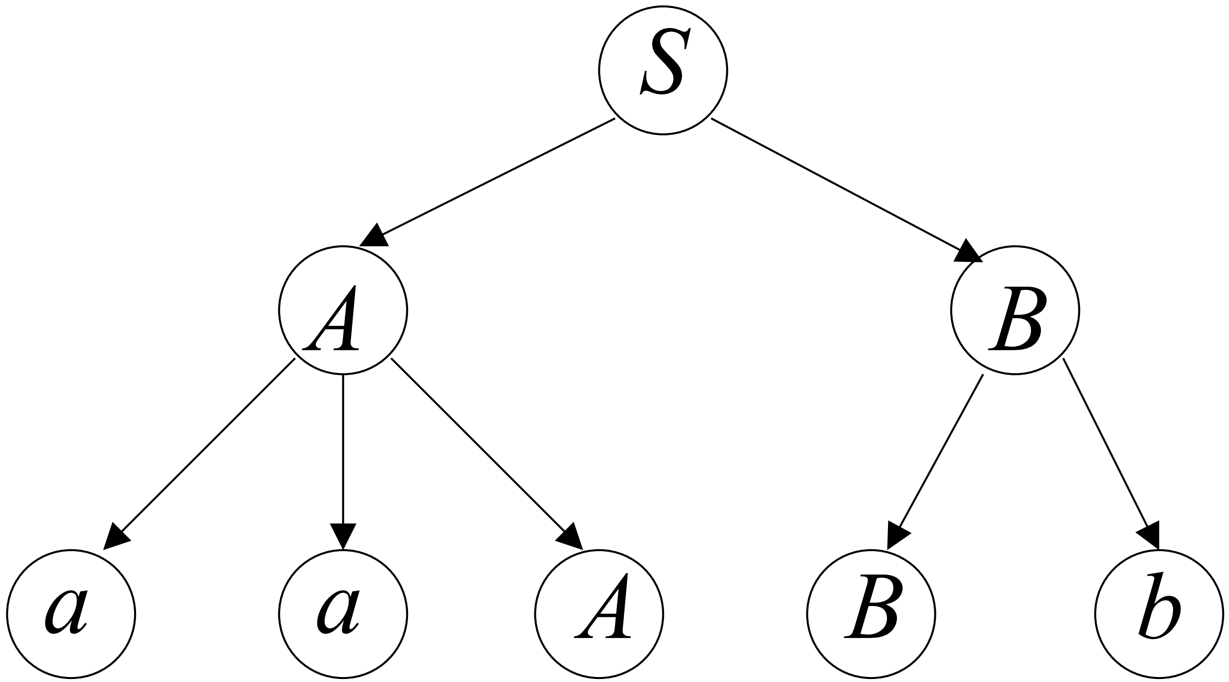
$$S \to AB \qquad A \to aaA \mid \lambda \qquad B \to Bb \mid \lambda$$

$$S \Rightarrow AB \Rightarrow aaAB$$



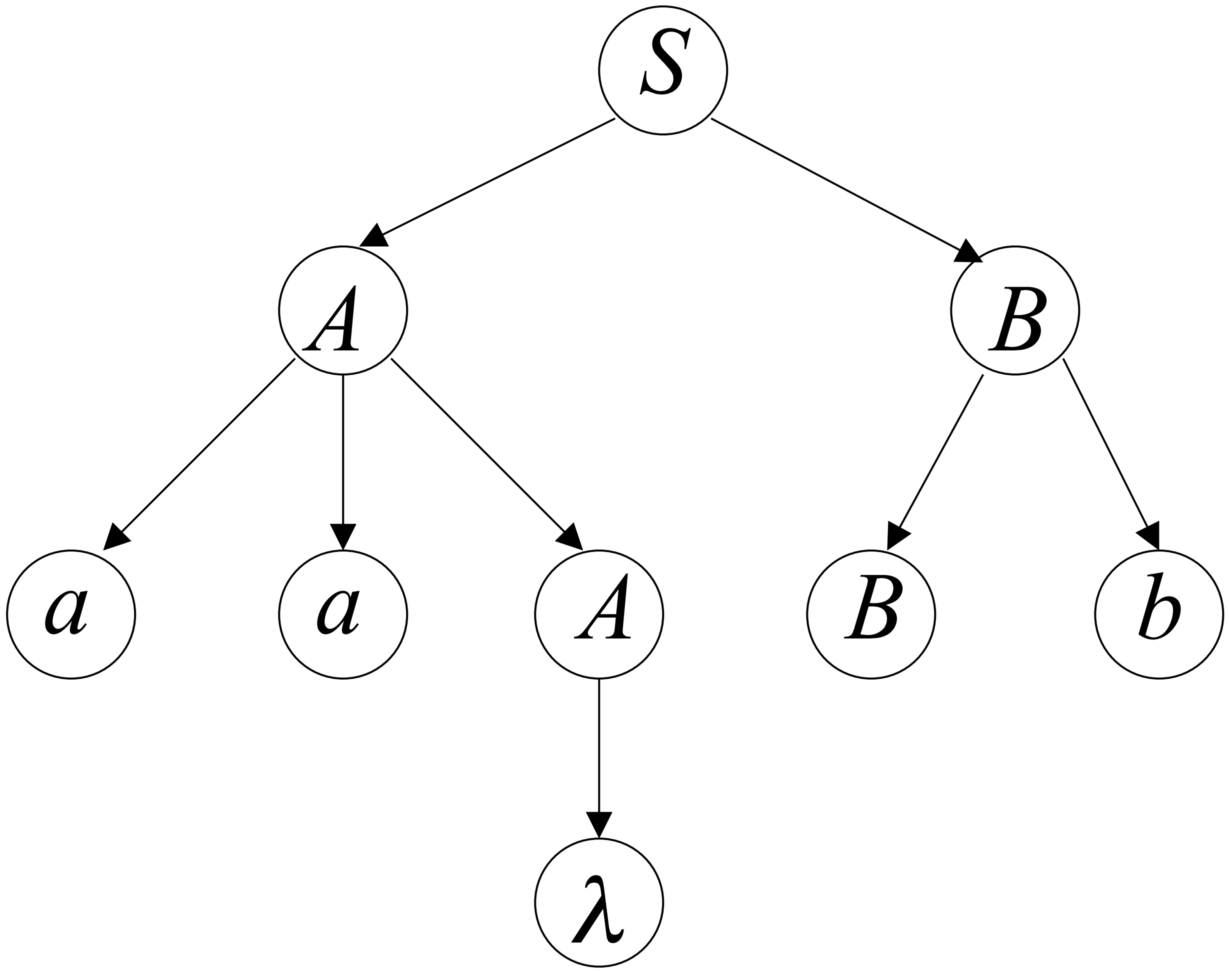
$$S \to AB \qquad A \to aaA \mid \lambda \qquad B \to Bb \mid \lambda$$

$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaABb$$



$$S \to AB \qquad A \to aaA \mid \lambda \qquad B \to Bb \mid \lambda$$

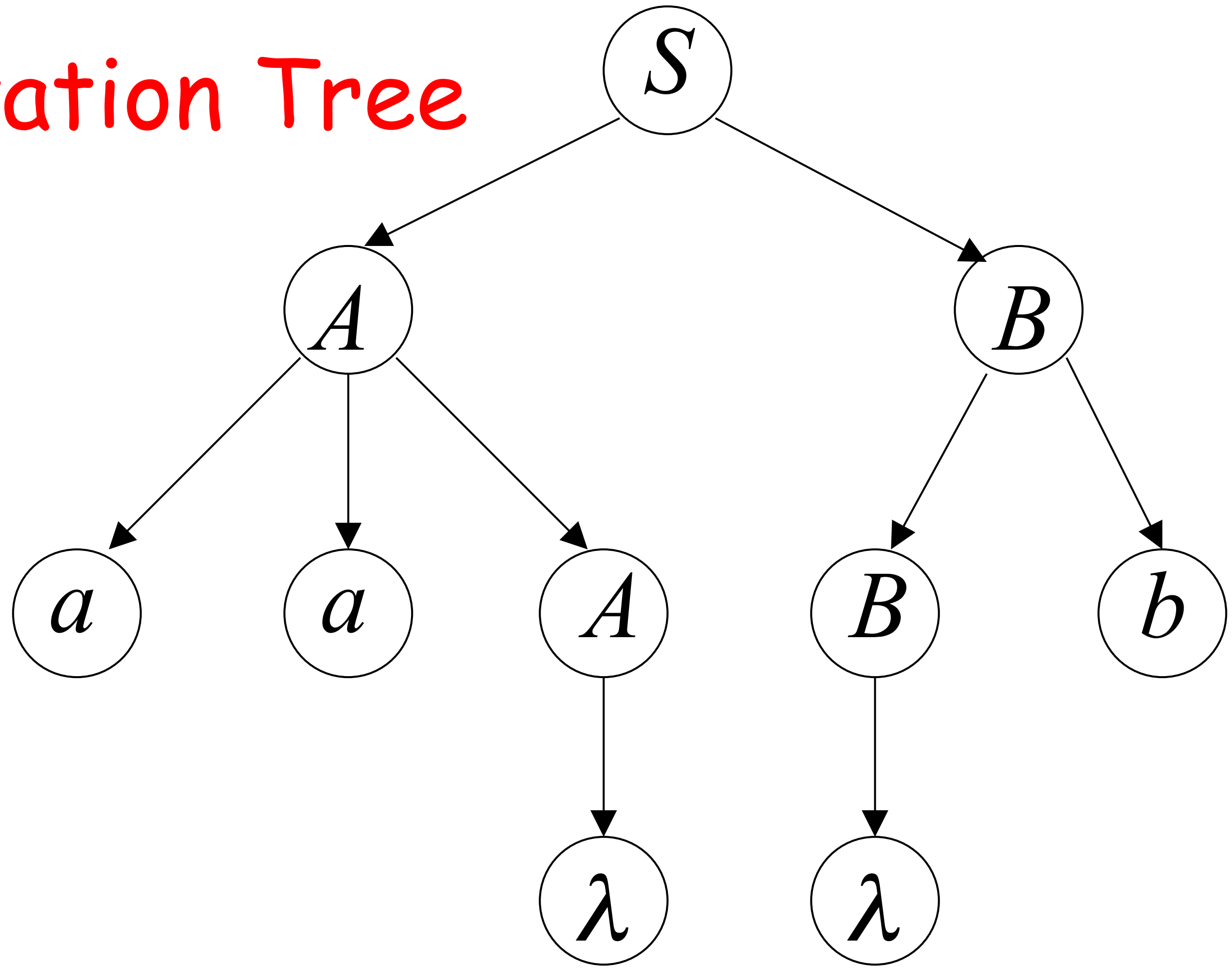
$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaABb \Rightarrow aaBb$$



$$S \to AB \qquad A \to aaA \mid \lambda \qquad B \to Bb \mid \lambda$$

$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaABb \Rightarrow aaBb \Rightarrow aab$$

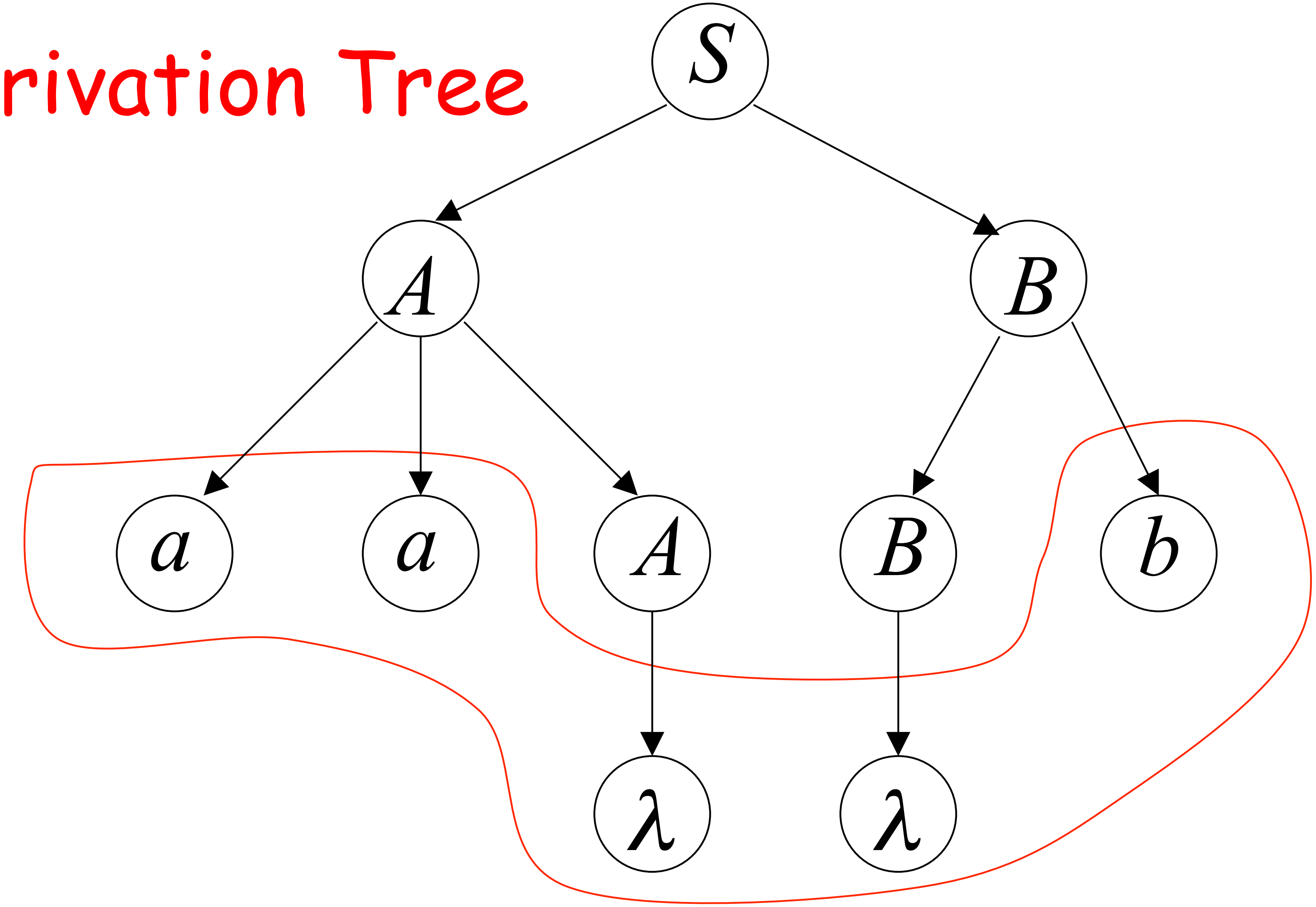
Derivation Tree



$$S \to AB \qquad A \to aaA \mid \lambda \qquad B \to Bb \mid \lambda$$

$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaABb \Rightarrow aaBb \Rightarrow aab$$

Derivation Tree



yield

$aa\lambda\lambda b$

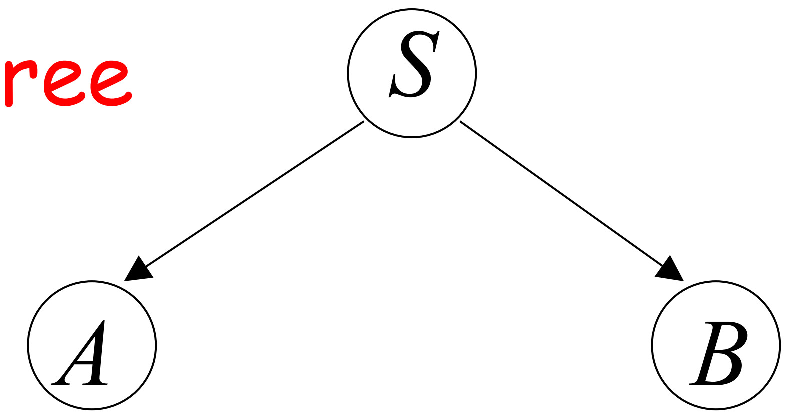
$= aab$

Partial Derivation Trees

$S \to AB \qquad A \to aaA \mid \lambda \qquad B \to Bb \mid \lambda$

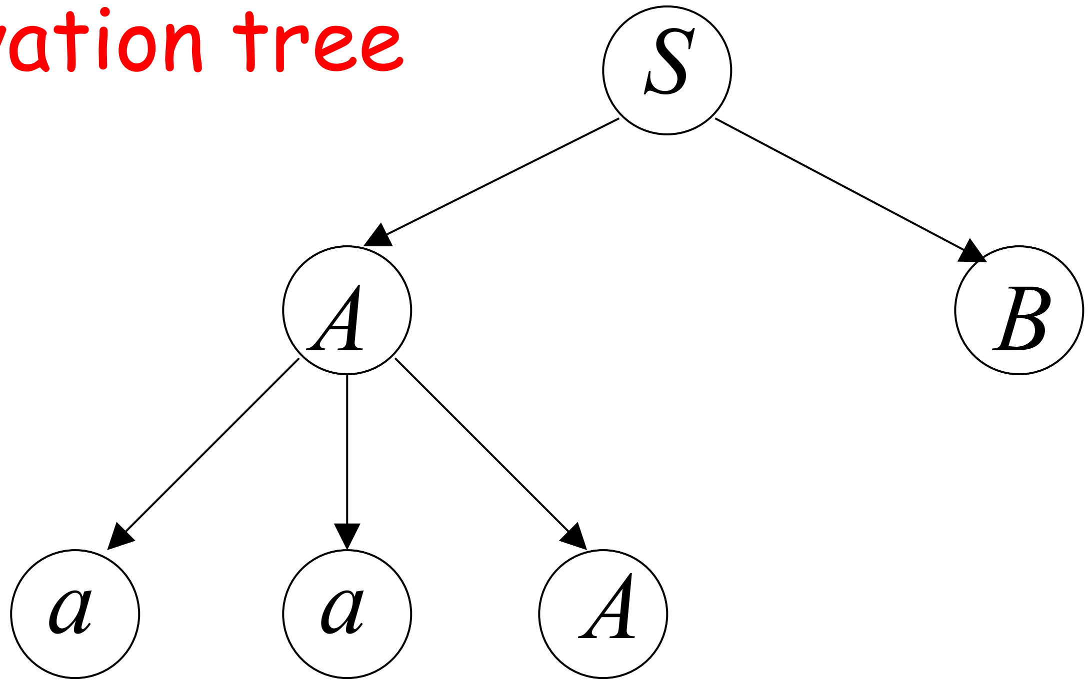
$S \Rightarrow AB$

Partial derivation tree



$$S \Rightarrow AB \Rightarrow aaAB$$

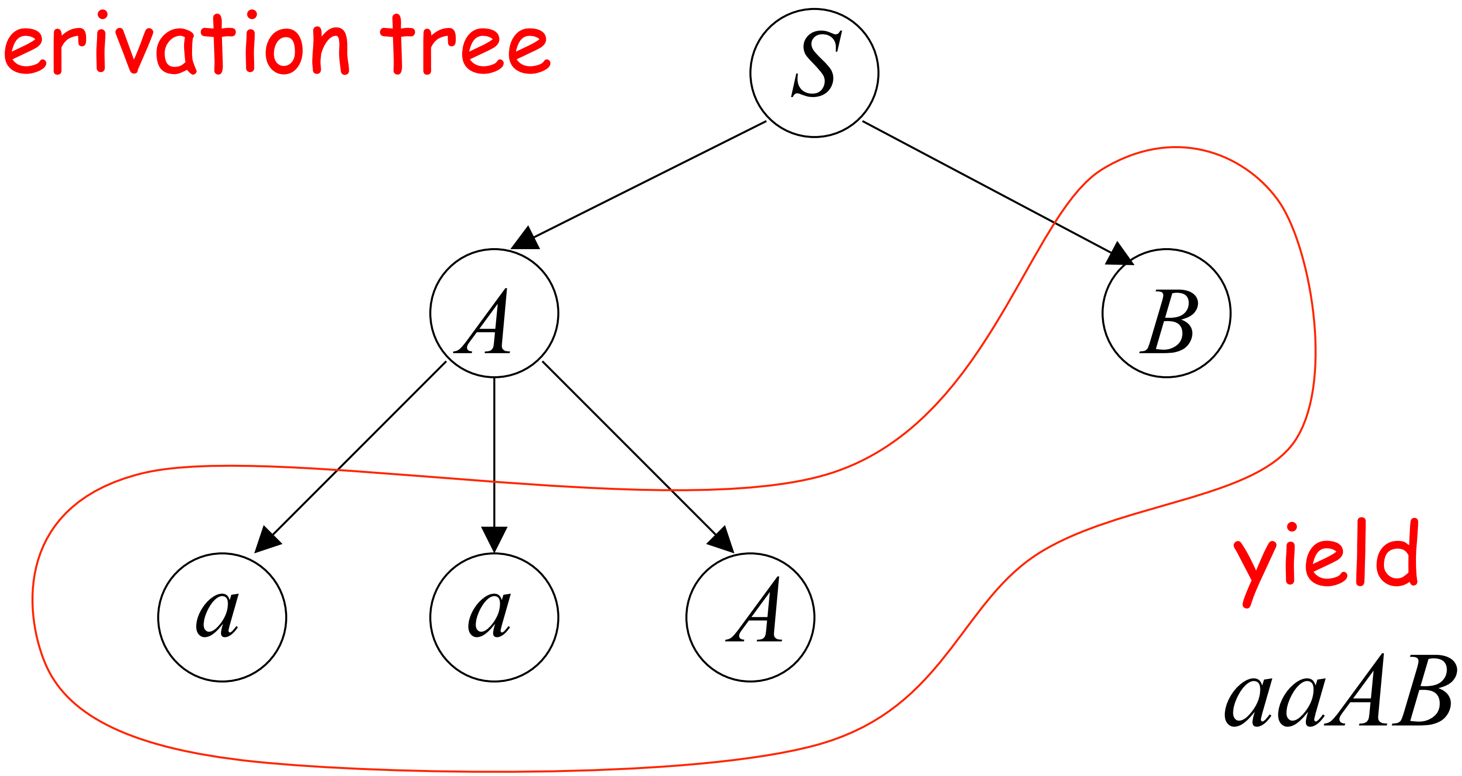
Partial derivation tree



$$S \Rightarrow AB \Rightarrow aaAB$$

sentential form

Partial derivation tree



Sometimes, derivation order doesn't matter

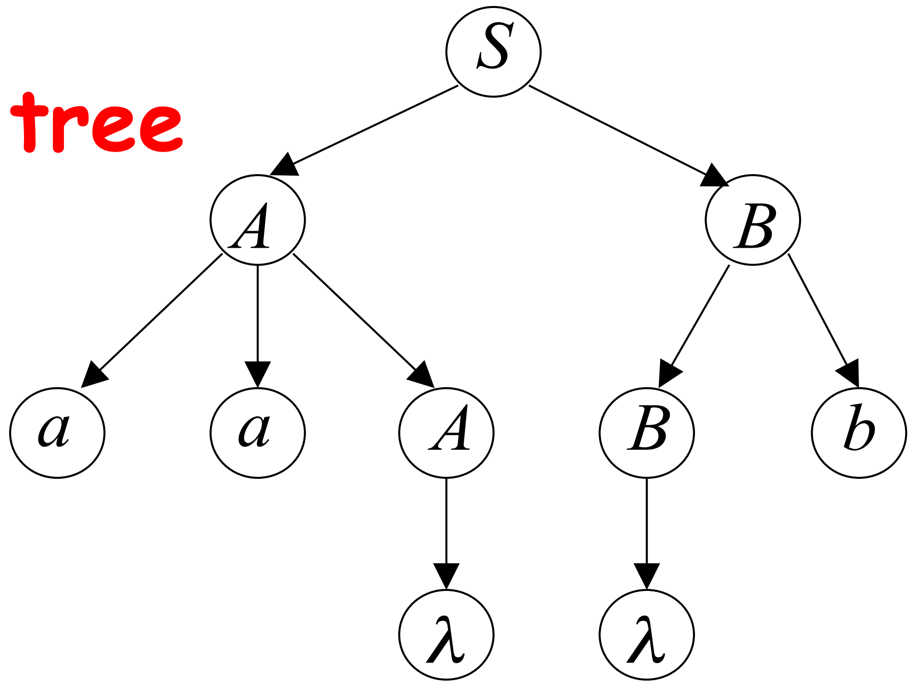
Leftmost:

$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaB \Rightarrow aaBb \Rightarrow aab$$

Rightmost:

$$S \Rightarrow AB \Rightarrow ABb \Rightarrow Ab \Rightarrow aaAb \Rightarrow aab$$

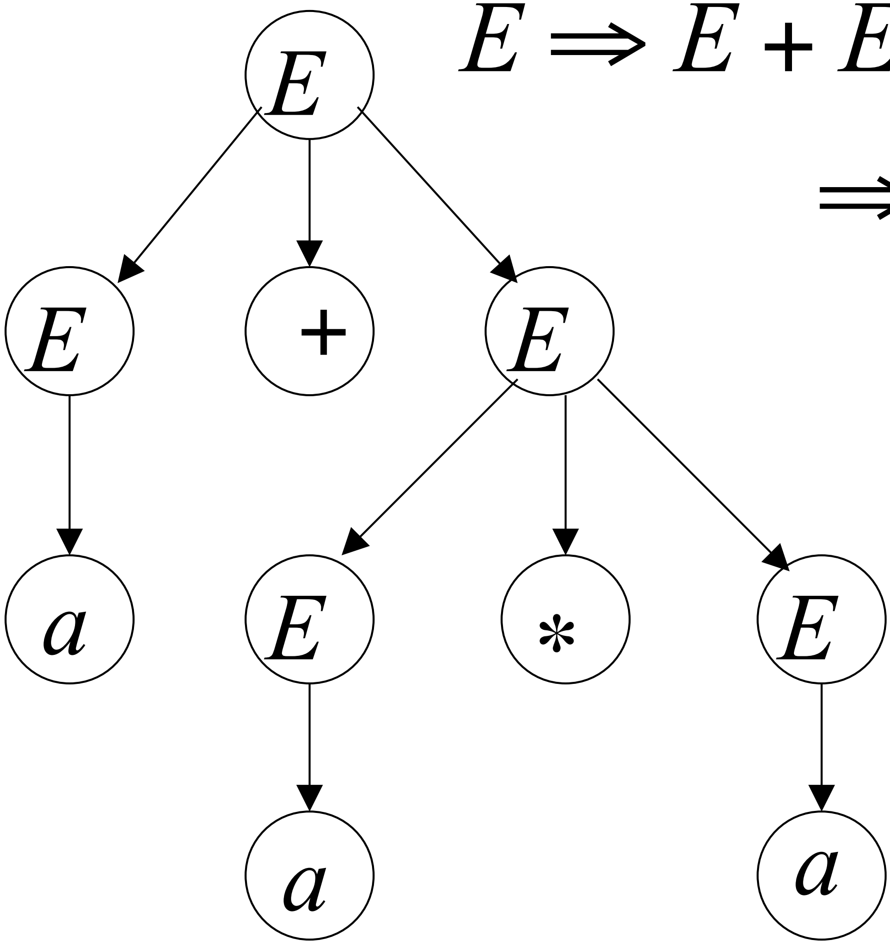
Same derivation tree



Ambiguity

$$E \rightarrow E + E \;\;|\;\; E * E \;\;|\;\; (E) \;\;|\;\; a$$

$$a + a * a$$



$$E \Rightarrow E + E \Rightarrow a + E \Rightarrow a + E * E$$

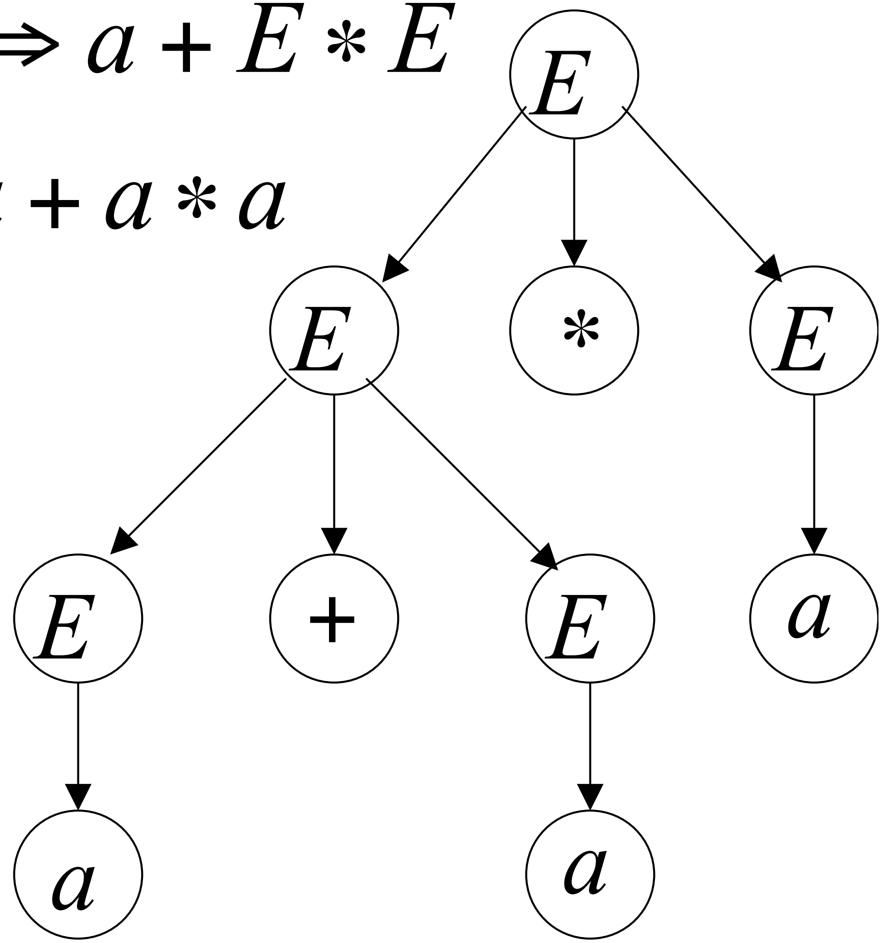
$$\Rightarrow a + a * E \Rightarrow a + a * a$$

leftmost derivation

$$E \rightarrow E + E \mid E * E \mid (E) \mid a$$

$$a + a * a$$

$$E \Rightarrow E * E \Rightarrow E + E * E \Rightarrow a + E * E$$
$$\Rightarrow a + a * E \Rightarrow a + a * a$$

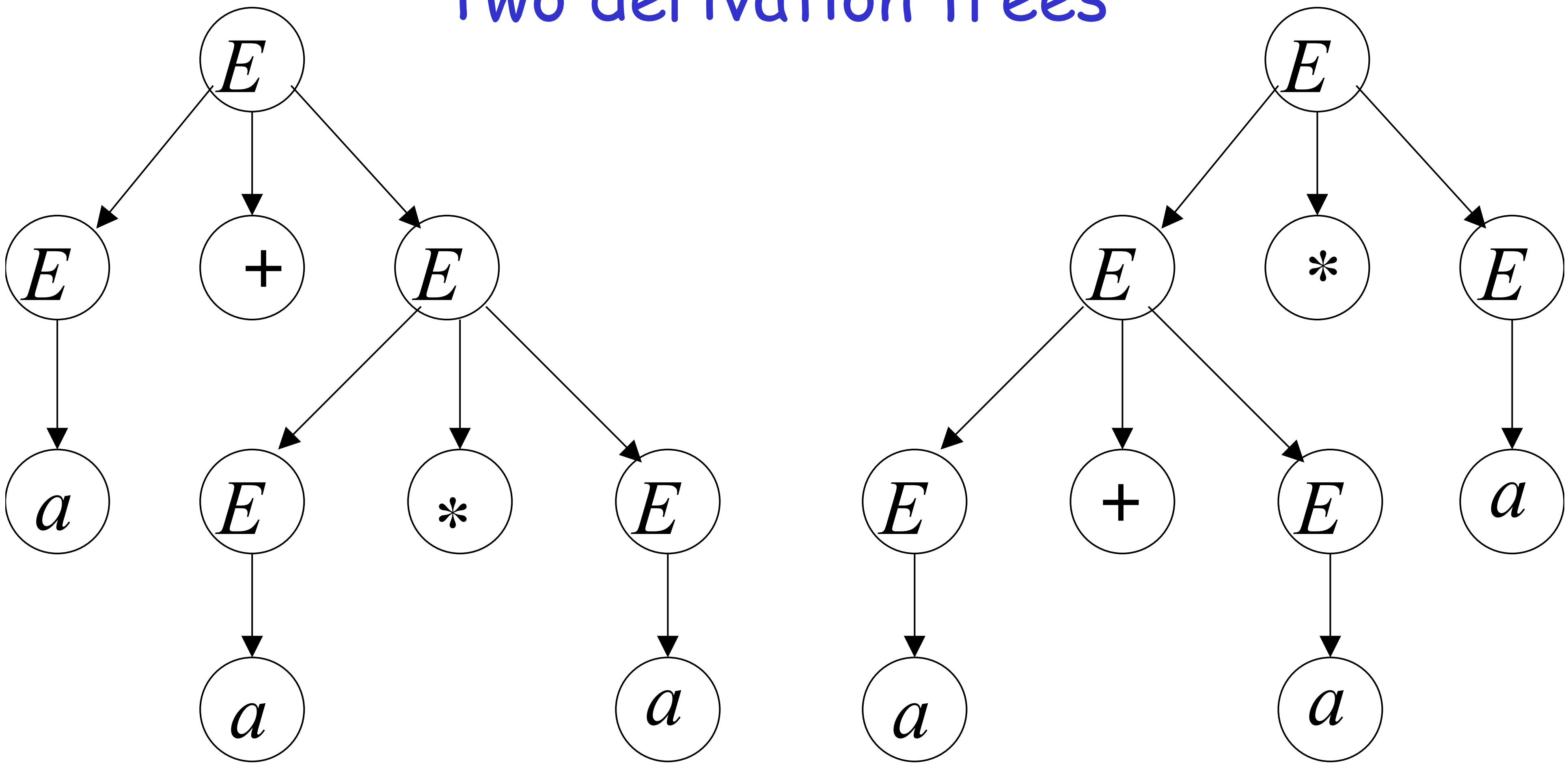


leftmost derivation

$$E \to E + E \;\;|\;\; E * E \;|\; (E) \;|\; a$$

$$a + a * a$$

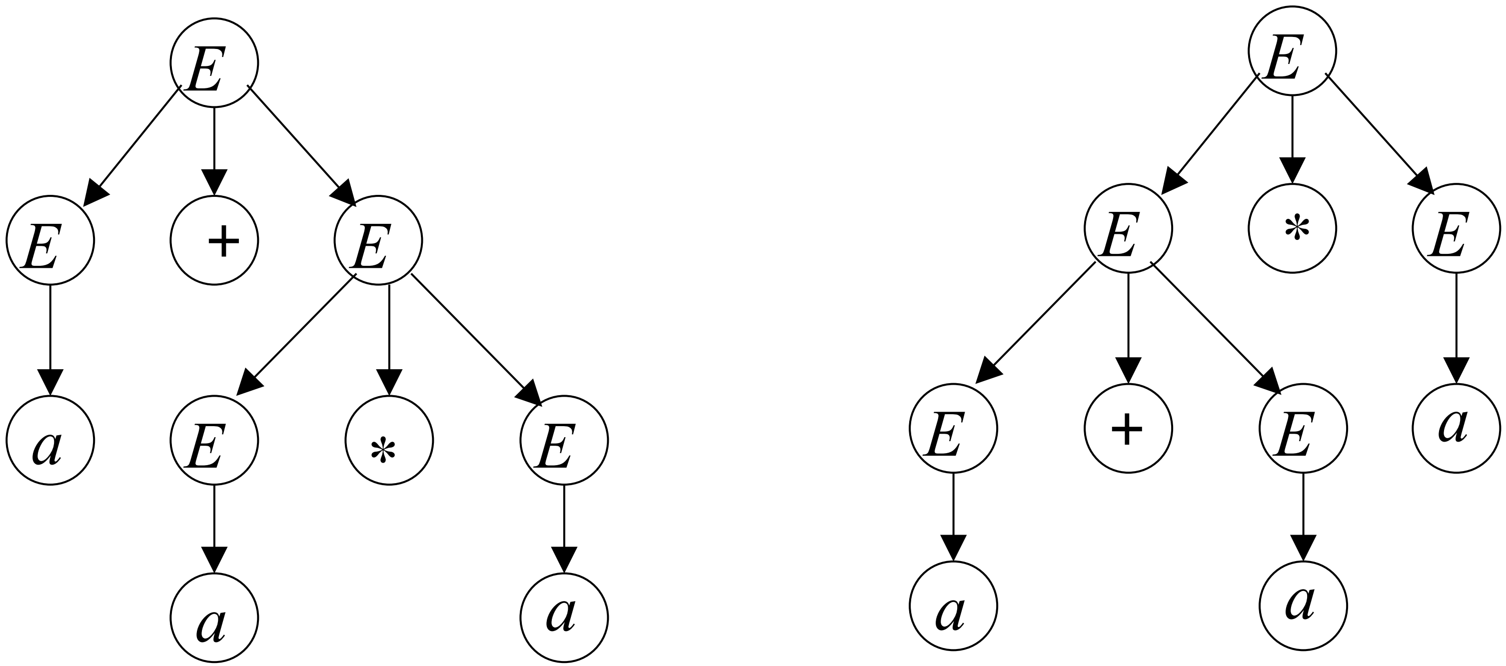
Two derivation trees



The grammar $E \to E + E \mid E * E \mid (E) \mid a$

is ambiguous:

string $a + a * a$ has two derivation trees



The grammar $E \rightarrow E + E \mid E * E \mid (E) \mid a$

is ambiguous:

string $a + a * a$ has two leftmost derivations

$$E \Rightarrow E + E \Rightarrow a + E \Rightarrow a + E * E$$
$$\Rightarrow a + a * E \Rightarrow a + a * a$$

$$E \Rightarrow E * E \Rightarrow E + E * E \Rightarrow a + E * E$$
$$\Rightarrow a + a * E \Rightarrow a + a * a$$

Definition:

A context-free grammar G is **ambiguous**

if some string $w \in L(G)$ has:

two or more derivation trees

In other words:

A context-free grammar G is **ambiguous**

if some string $w \in L(G)$ has:

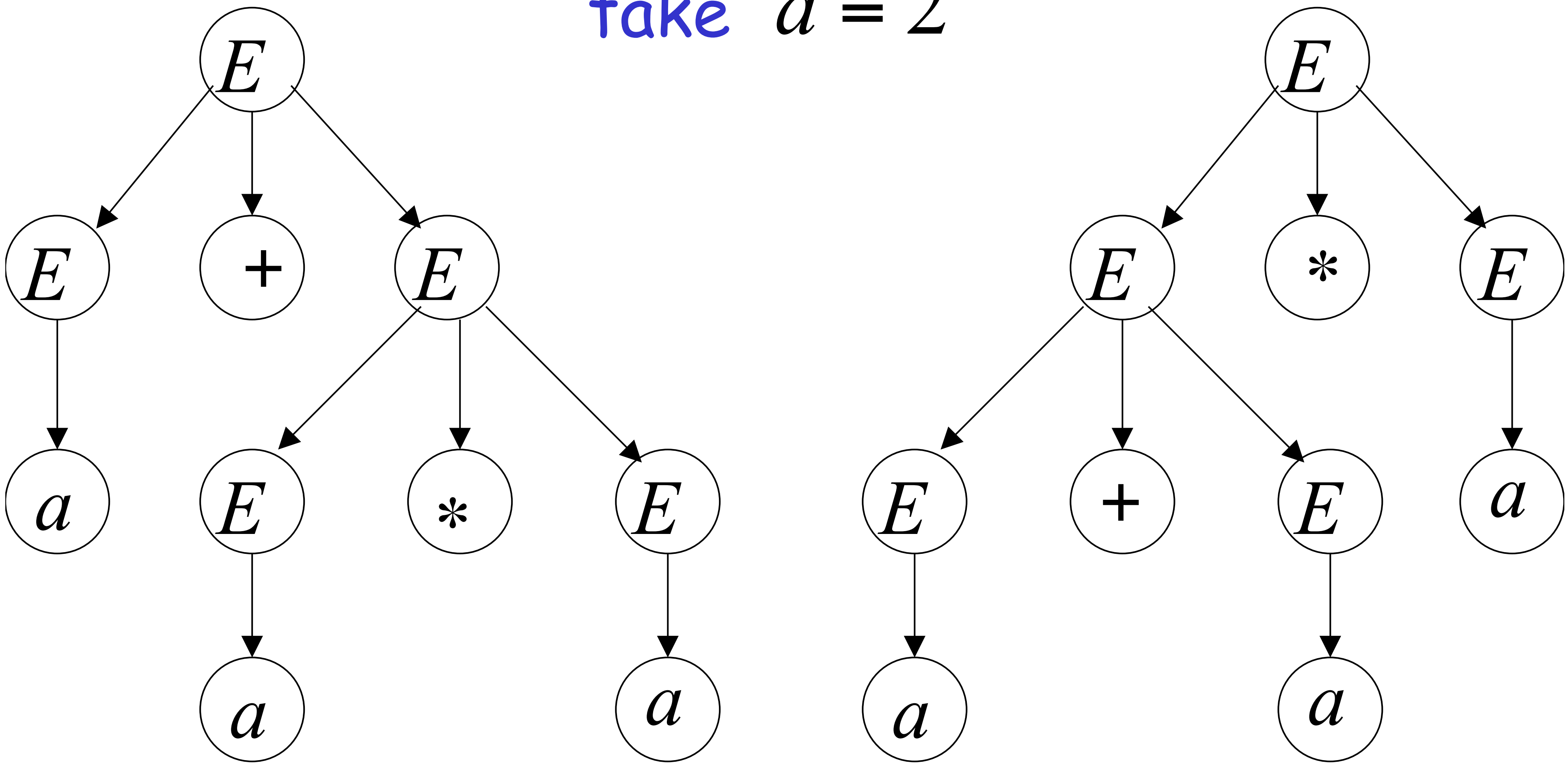
two or more leftmost derivations

(or rightmost)

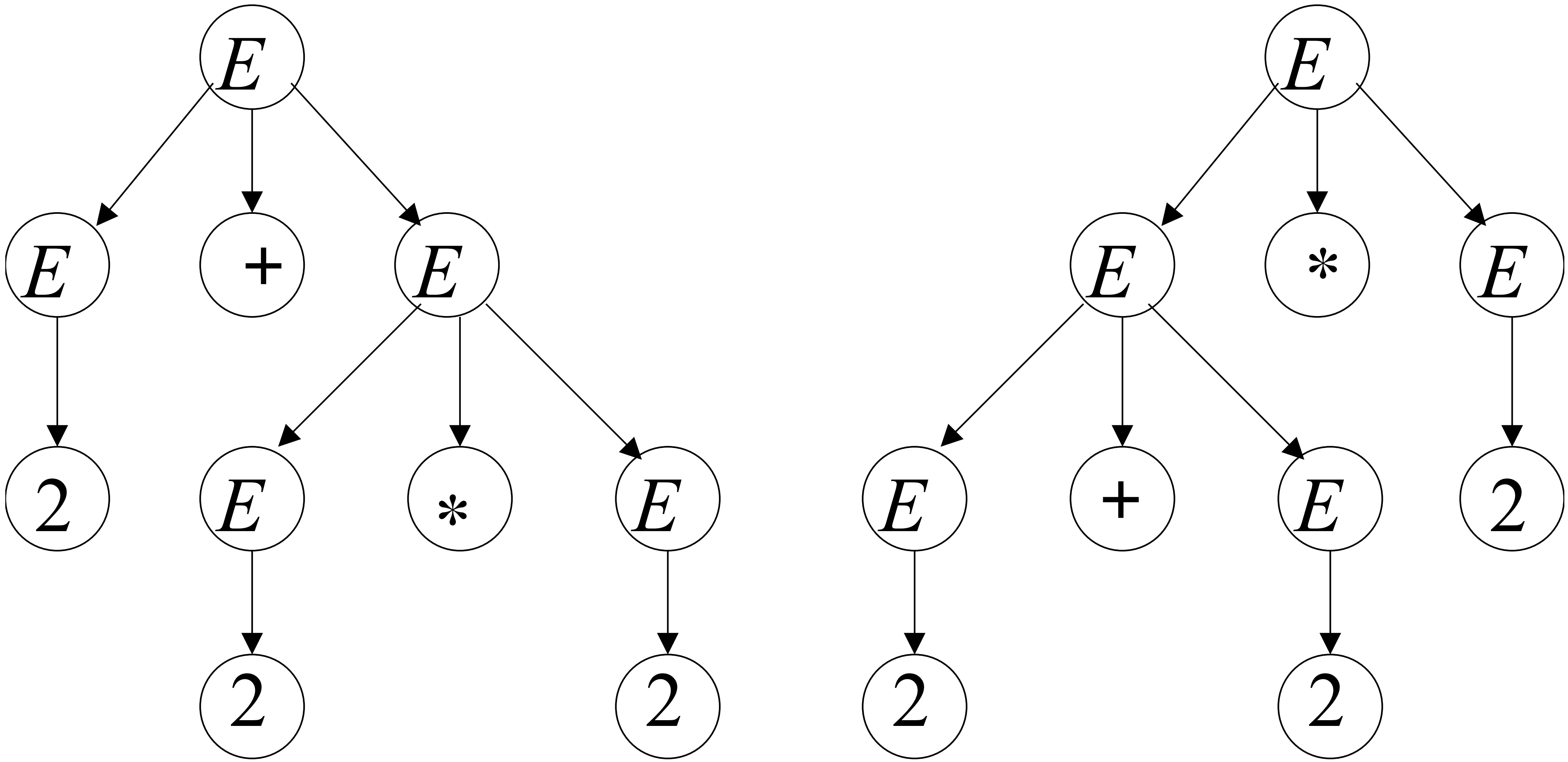
Why do we care about ambiguity?

$$a + a * a$$

take $a = 2$

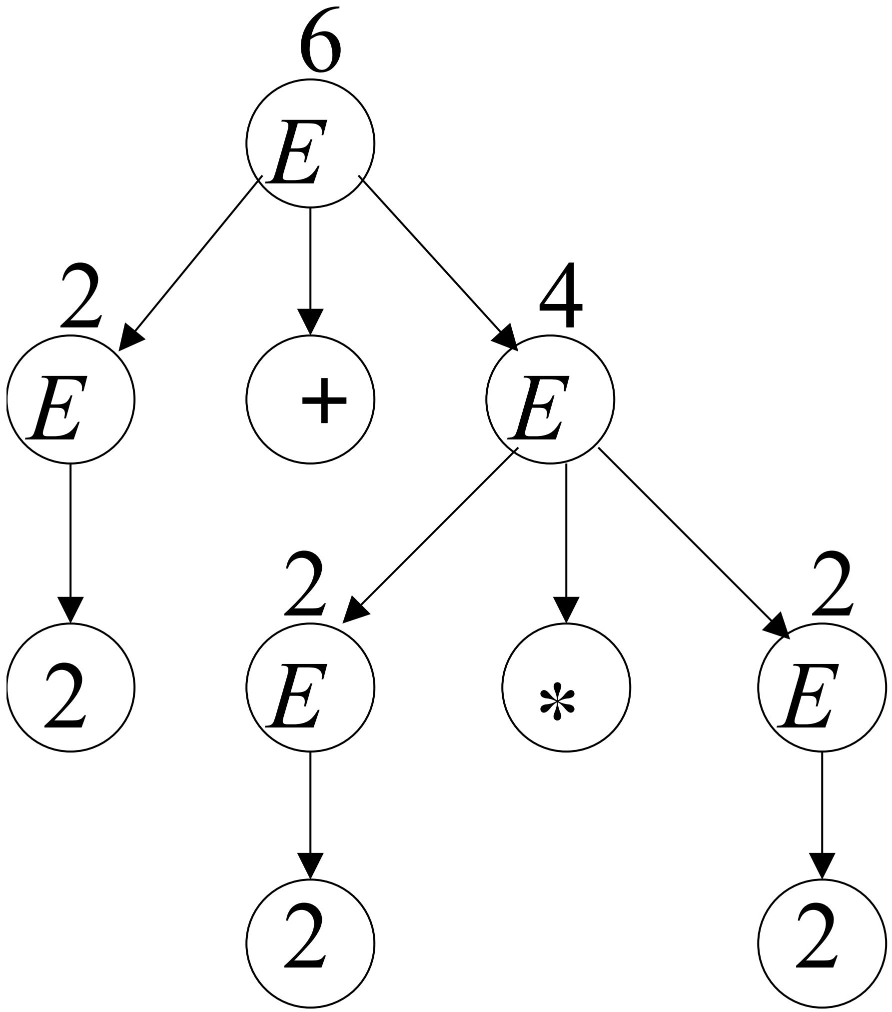


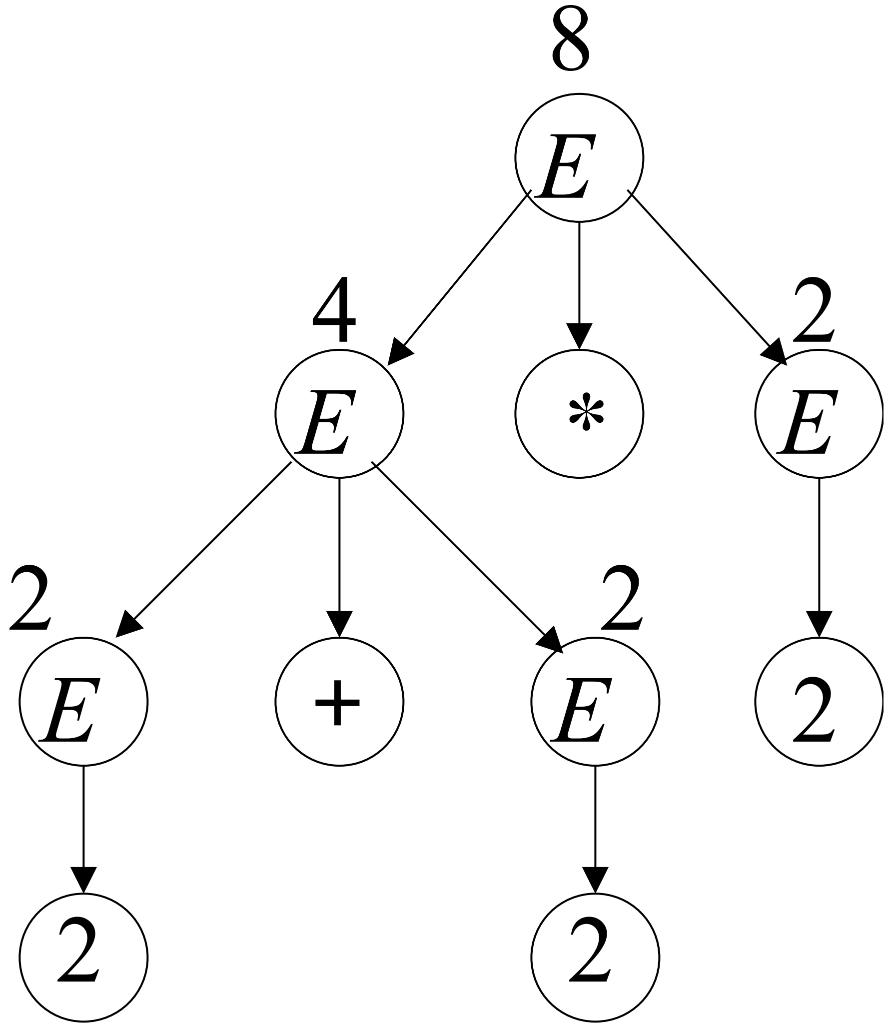
$$2 + 2 * 2$$



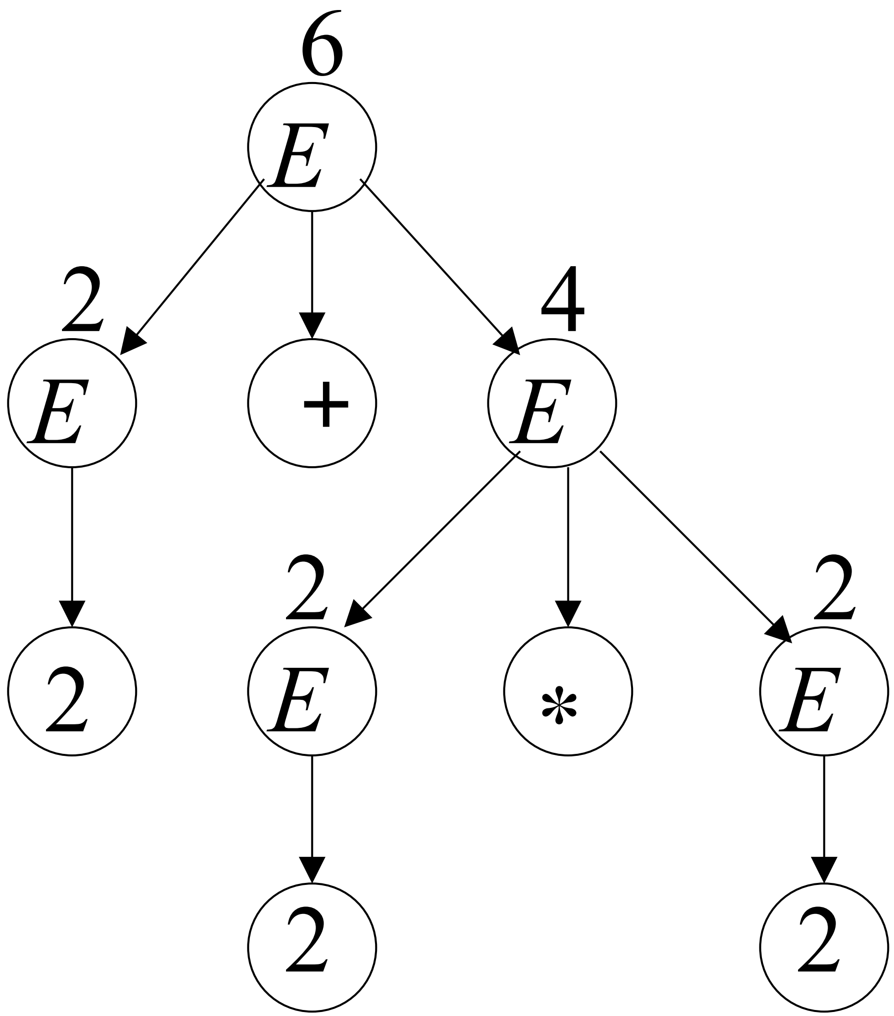
$$2 + 2 * 2 = 6$$

$$2 + 2 * 2 = 8$$





Correct result: $2+2*2=6$



- Ambiguity is **bad** for programming languages

- We want to remove ambiguity

We fix the ambiguous grammar:

$$E \to E + E \;\;|\; E * E \;|\; (E) \;|\; a$$

New non-ambiguous grammar:

$$E \to E + T$$

$$E \to T$$

$$T \to T * F$$

$$T \to F$$

$$F \to (E)$$

$$F \to a$$

$$E \Rightarrow E + T \Rightarrow T + T \Rightarrow F + T \Rightarrow a + T \Rightarrow a + T * F$$
$$\Rightarrow a + F * F \Rightarrow a + a * F \Rightarrow a + a * a$$

$E \rightarrow E + T$

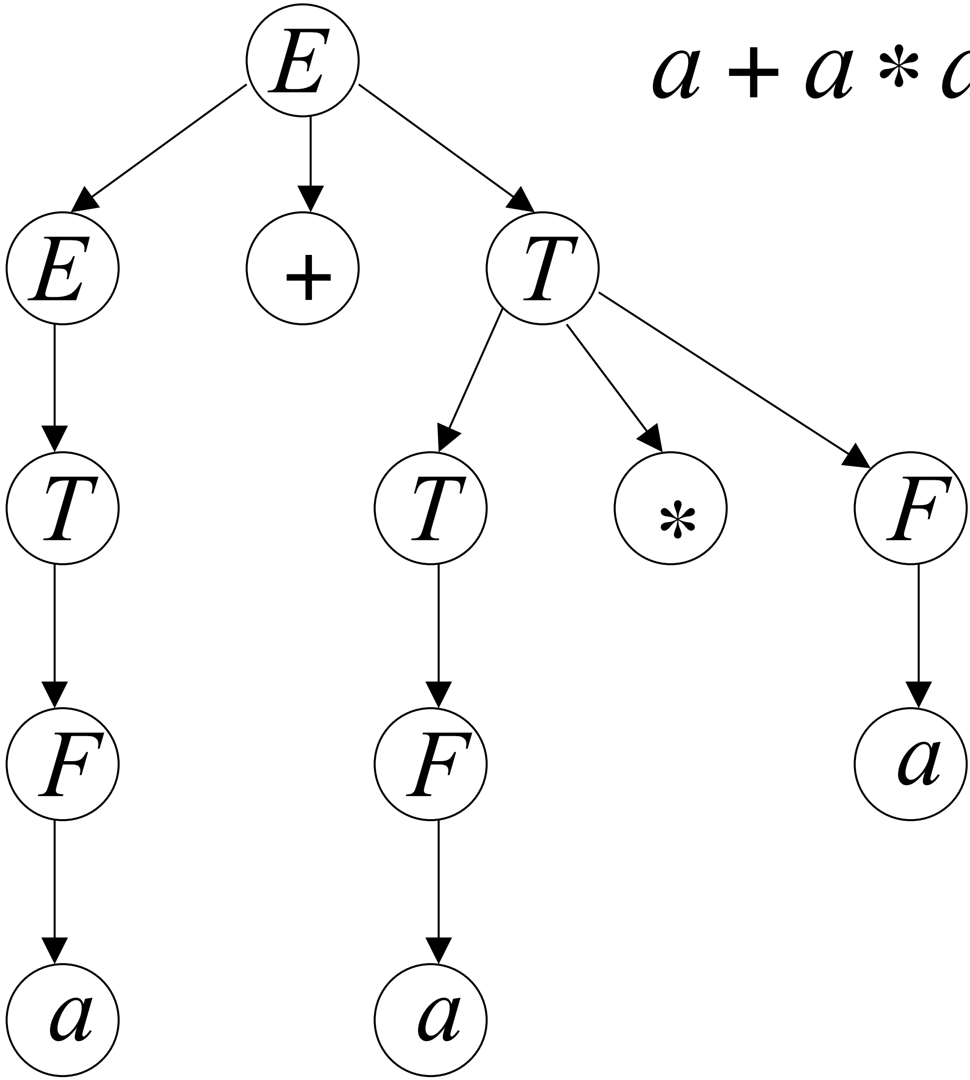
$E \rightarrow T$

$T \rightarrow T * F$

$T \rightarrow F$

$F \rightarrow (E)$

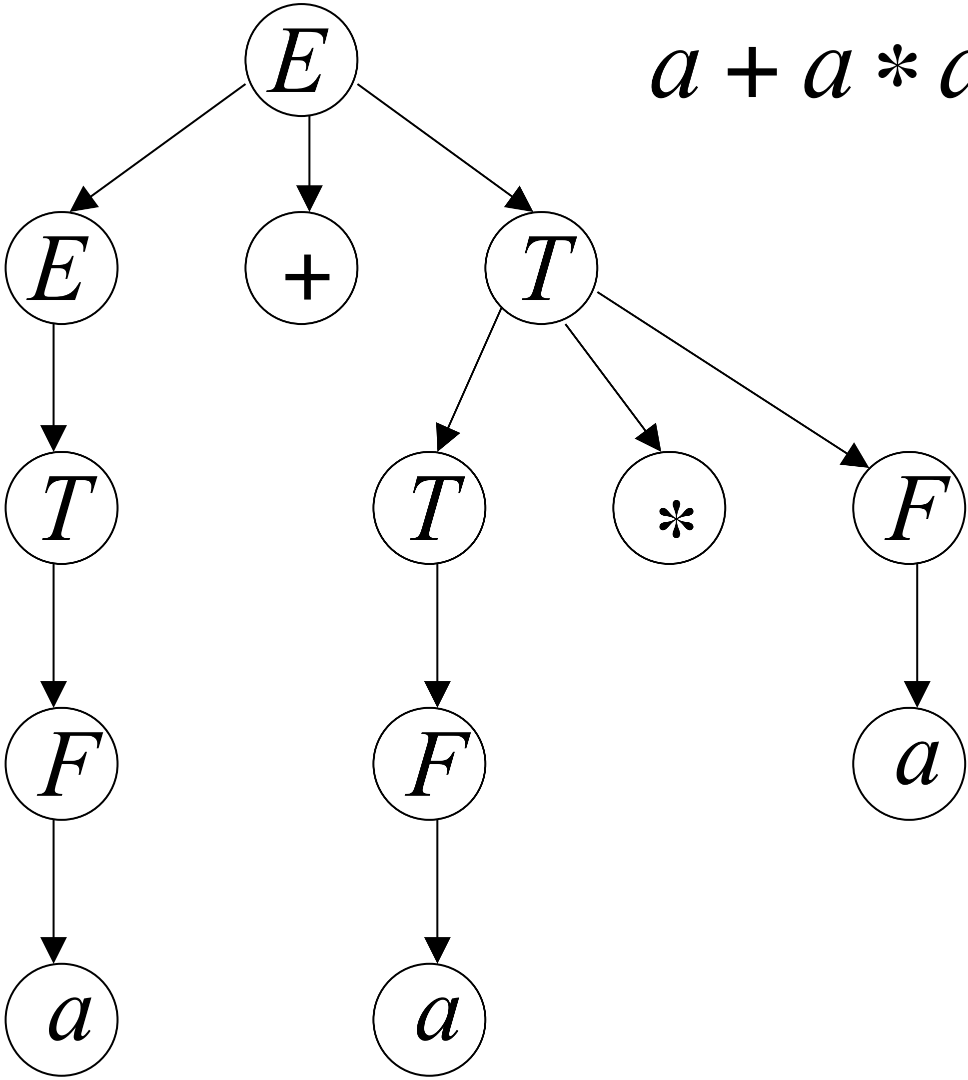
$F \rightarrow a$



$a + a * a$

Unique derivation tree

$a + a * a$



The grammar G:

$$E \to E + T$$

$$E \to T$$

$$T \to T * F$$

$$T \to F$$

$$F \to (E)$$

$$F \to a$$

is non-ambiguous:

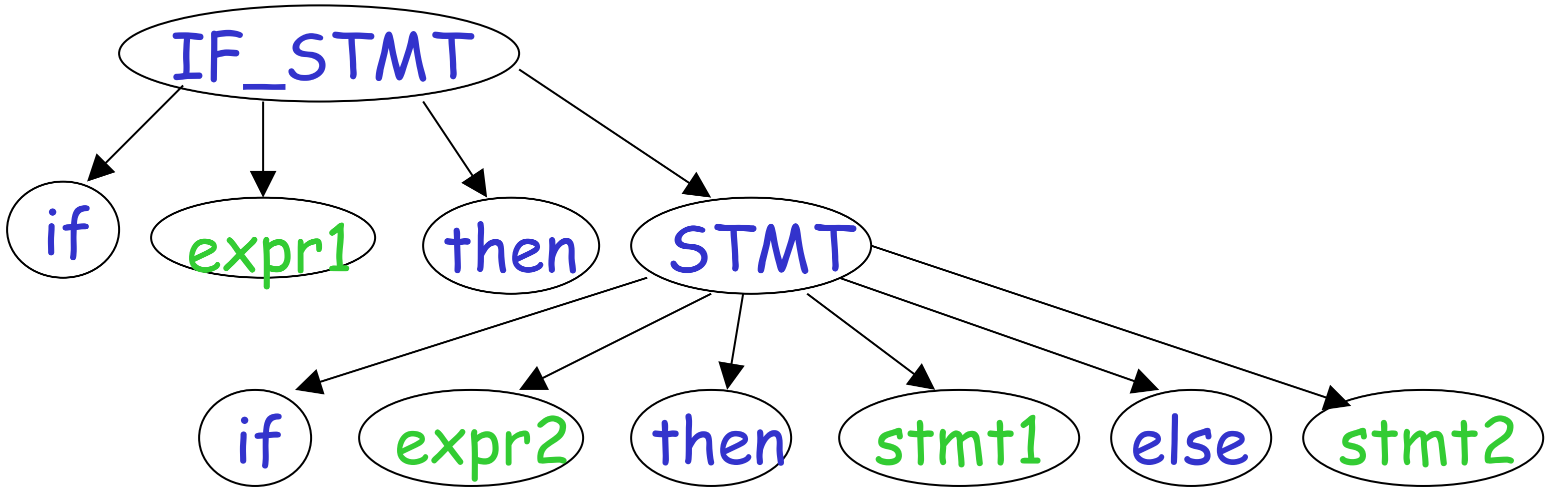
Every string $w \in L(G)$ has a unique derivation tree

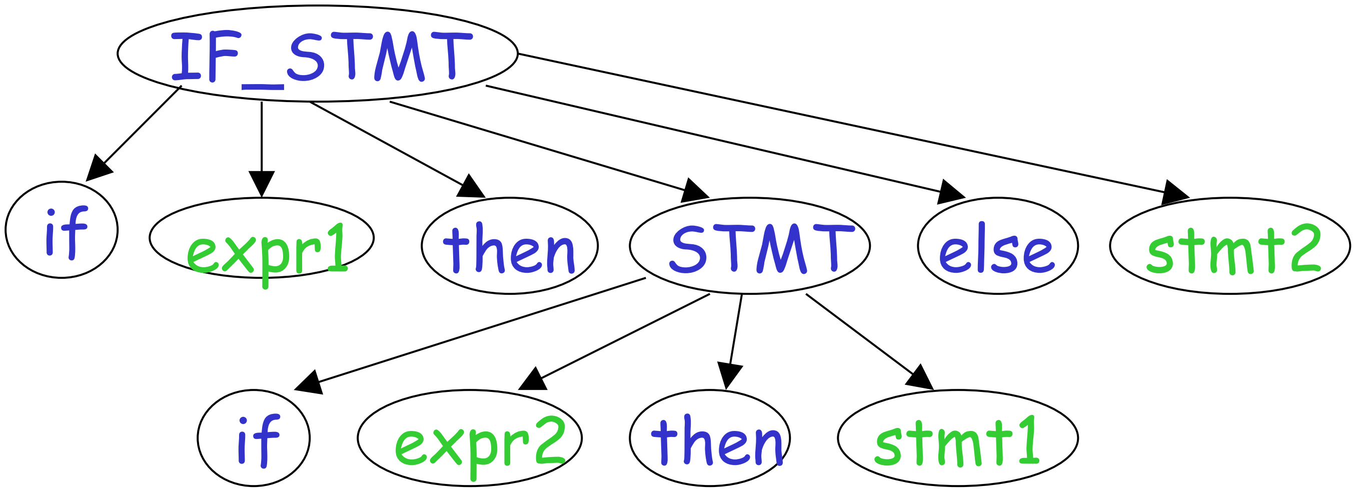
Another Ambiguous Grammar

IF_STMT → if EXPR then STMT
| if EXPR then STMT else STMT

Ambiguity?

If expr1 then if expr2 then stmt1 else stmt2





Inherent Ambiguity

Some context free languages
have only ambiguous grammars

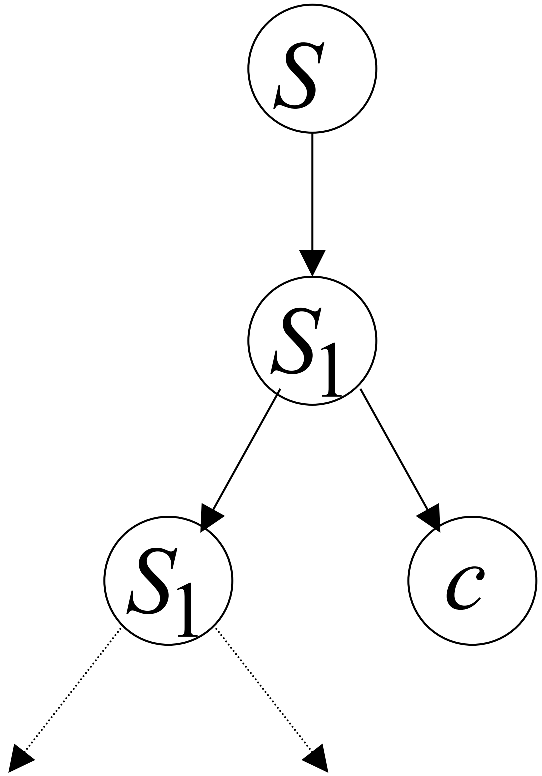
Example: $L = \{a^n b^n c^m\} \cup \{a^n b^m c^m\}$

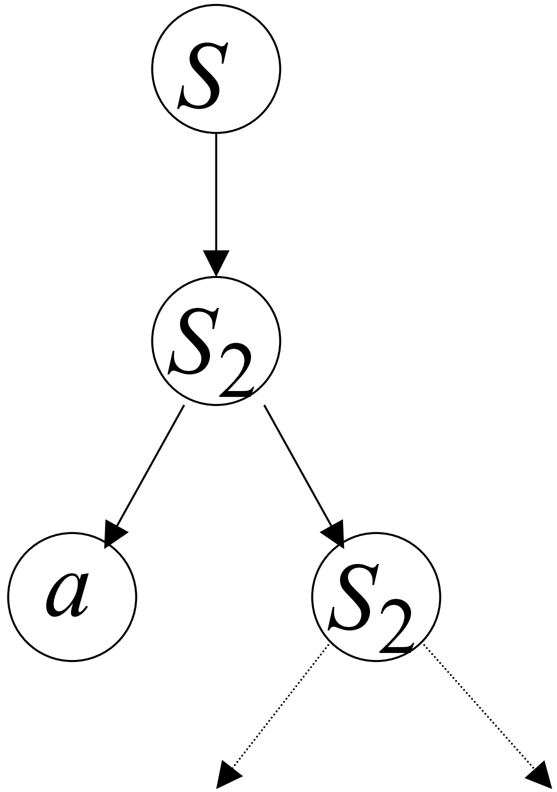
$$S \to S_1 \mid S_2 \qquad S_1 \to S_1 c \mid A \qquad S_2 \to a S_2 \mid B$$

$$A \to aAb \mid \lambda \qquad B \to bBc \mid \lambda$$

The string $a^n b^n c^n$

has two derivation trees





Ambiguity in natural language?

Take-away

Definition context-free grammar

Definition context-free language

Derivation, sentential form, sentence

Derivation trees

Ambiguity

Context-free grammars for natural language