C-Dioids and μ-Continuous Chomsky-Algebras

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Abstract. We prove that the categories of $\mathcal{C}$-dioids of Hopkins 2008 and of μ-continuous Chomsky-algebras of Grathwohl, Henglein and Kozen 2013 are the same.

1 Introduction

The equational theory of the class of context-free languages has been axiomatized in 2013 by Grathwohl, Henglein and Kozen [3], using μ-terms as notation system for context-free grammars. In order to be able to interpret μ as a least fixed point operator, they consider algebraically closed idempotent semirings, called "Chomsky algebras". An idempotent semiring $(M, +, \cdot, 0, 1)$ is algebraically closed if every finite system of inequations $x_1 \geq p_1(x_1, \ldots, x_n), \ldots, x_n \geq p_n(x_1, \ldots, x_n)$, where $p_1, \ldots, p_n$ are polynomials, has a least solution. The essential part of their axiomatization is the μ-continuity axiom, relating $\cdot$ with the least fixed point operator μ and a least upper bound operation $\sum$:

$$a \cdot \mu x t \cdot b = \sum \{\, a \cdot m x t \cdot b \mid m \in \mathbb{N} \,\},$$

where $a, b \in M$ and mxt is the m-fold iteration of the map $x \mapsto t$.

To give an algebraic generalization of the Chomsky hierarchy of language classes to classes of subsets of monoids, Hopkins [6] has introduced "monadic operators" $\mathcal{A}$ that assign to each monoid M a set $\mathcal{A}M \subseteq \mathcal{P}M$ of its subsets; $\mathcal{A}M$ is assumed to contain all finite subsets of M, so $(\mathcal{A}M, \cup, \cdot, \emptyset, \{1\})$ is an idempotent semiring or "dioid" in the terminology of [4, 6]. An "$\mathcal{A}$-dioid" is a dioid $(M, +, \cdot, 0, 1)$ where (i) each $U \in \mathcal{A}M$ has a least upper bound $\sum U \in M$ and (ii) for each $U, V \in \mathcal{A}M$, $(\sum U) \cdot (\sum V) = \sum (U \cdot V)$. This can also be seen as a continuity assumption for $\cdot$, and, assuming (i), is equivalent to:

$$a(\sum U)b = \sum (aUb), \qquad \text{for all } U \in \mathcal{A}M, a, b \in M.$$

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We here show that if $\mathcal{A}$ is specialized to the operator $\mathcal{C}$ that returns the context-free subsets of monoids, then the $\mathcal{C}$-dioids and the μ-continous Chomsky-algebras are the same. Since both approaches give a notion of context-freeness, this is to be be expected, but had been left unresolved in Leiß [11].

Both definitions have their advantages and drawbacks. The main advantage of the μ-continuous Chomsky algebras is, of course, that they lead to an infinitary axiomatization of the equational theory of context-free langugages. A disadvantage is that μ-terms are just formal solution terms and, when nested, intuitively incomprehensible; moreover, the context-free subsets are kept in the background. An advantage of the $\mathcal{C}$-dioids is that they bring the context-free subsets in front and separate a completeness property of the partial order from the sup-continuity of the product; this allows for algebraic constructions like coproduct, coequalizer and tensor product in fairly standard ways that extend to other classes of dioids (cf. Hopkins and Leiß [8]). A drawback of the notion of $\mathcal{C}$-dioid may be its universal second-order formulation, which hides an equivalent formulation by infinitary equational implications.

2 Monadic Operators and Language Classes

Let $\mathbb{M}$ be the category of monoids $(M, \cdot, 1)$ and homomorphisms between monoids. A *partially ordered monoid* $(M, \cdot, 1, \leq)$ is a monoid $(M, \cdot, 1)$ with a partial order $\leq \subseteq M \times M$, with respect to which $\cdot$ is monotone in each argument.

A *semiring* $R = (R, +, 0, \cdot, 1)$ is a set R with two operations $+, \cdot : R \times R \to R$, such that $(R, +, 0)$ and $(R, \cdot, 1)$ are monoids, $+$ is commutative, and the zero and distributivity laws holds:

$$\forall a, b, c, d : \quad a0b = 0, \quad a(b + c)d = abd + acd.$$

A *dioid* or *idempotent semiring* $D = (D, +, 0, \cdot, 1)$ is a semiring in which $+$ is idempotent. Each dioid D has a natural partial order $\leq$, defined by

$$a \leq b :\iff a + b = b.$$

Let $\mathbb{D}$ be the category of dioids and dioid-homomorphisms.

If $M = (M, \cdot^M, 1^M)$ is a monoid, its power set $\mathcal{P}M$ is a partially ordered monoid $(\mathcal{P}M, \cdot, 1, \subseteq)$, using

$$A \cdot B := \{\, a \cdot^M b \mid a \in A, b \in B \,\} \quad \text{and} \quad 1 := \{1^M\},$$

and a dioid $(\mathcal{P}M, +, \cdot, 0, 1)$, using $A + B := A \cup B$ and $0 := \emptyset$.

We review some definitions of Hopkins [6]. A *monadic operator* $\mathcal{A}$ is a functor $\mathcal{A} : \mathbb{M} \to \mathbb{D}$ such that for each monoid M

A_0) $\mathcal{A}M$ is a set of subsets of M,
A_1) $\mathcal{A}M$ contains each finite subset of M,
A_2) $\mathcal{A}M$ is closed under product (hence a monoid),
A_3) $\mathcal{A}M$ is closed under union of sets from $\mathcal{A}\mathcal{A}M$ (hence a dioid), and

A_4) $\mathcal{A}$ preserves monoid-homomorphisms: if $f : M \to N$ is a homomorphism, so is $\mathcal{A}f : \mathcal{A}M \to \mathcal{A}N$, where for $U \subseteq M$

$$\mathcal{A}f(U) := \{\, f(u) \mid u \in U \,\}.$$

For brevity, we often write $\widetilde{f}$ instead of $\mathcal{A}f$.

Remark 1. Each monadic operator $\mathcal{A}$ gives rise to a subcategory $\mathbb{D}\mathcal{A}$ of $\mathbb{D}$ and is the left adjoint of an adjunction $(\mathcal{A}, \widehat{\mathcal{A}}, \eta, \epsilon) : \mathbb{M} \to \mathbb{D}\mathcal{A}$ (cf. Mac Lane [13]), where $\widehat{\mathcal{A}} : \mathbb{D}\mathcal{A} \to \mathbb{M}$ is the forgetful functor and for $M \in \mathbb{M}$, $\eta_M(m) = \{m\} \in \mathcal{A}M$, and $\epsilon_M : \mathcal{A}M \to M$ maps $U \in \mathcal{A}M$ to its least upper bound $\sum U \in M$. This adjunction gives rise to a monad $T = (\widehat{\mathcal{A}} \circ \mathcal{A}, \eta, \mu)$, an endofunctor on $\mathbb{M}$, where the unit η is taken from the adjunction and the product μ maps $\mathcal{U} \in \mathcal{A}\mathcal{A}M$ to $\bigcup \mathcal{U} \in \mathcal{A}M$. While $\widehat{\mathcal{A}}$ is called a *monadic functor* in category theory (see [13], p. 139), Hopkins [6] calls $\mathcal{A}$ a *monadic operator*, a term we also use here.

Example 1. The power set operator $\mathcal{P}$ is a monadic operator. The operator $\mathcal{F}$ that assigns to M the set $\mathcal{F}M$ of all finite subsets of M is a monadic operator.

Example 2. The operator $\mathcal{R}$ that assigns to M the closure of $\mathcal{F}M$ under $+$ (union), $\cdot$ (elementwise product) and * (iteration, Kleene's star), is monadic. Hopkins [6] defines monadic operators $\mathcal{C}$ and $\mathcal{T}$ that select the context-free subsets $\mathcal{C}M$ and the Turing/Thue-subsets $\mathcal{T}M$ of a monoid.

We give a simpler, but equivalent definition of $\mathcal{C}$ in Section 3 below. A correction of the definition of $\mathcal{T}$ in [6] is given in Hopkins and Leiß [8].

Let M be a partially ordered monoid. For $a \in M$ and $U \subseteq M$ let $U < a$ mean that a is an upper bound of U, i.e. for all $u \in U$, $u \leq a$. M is $\mathcal{A}$-*complete*, if each $U \in \mathcal{A}M$ has a least upper bound $\sum U \in M$. M is $\mathcal{A}$-*distributive*, if for all $U, V \in \mathcal{A}M$, $\sum(UV) = \sum U \cdot \sum V$.

$\mathcal{A}$-distributivity states $\sum$-continuity of $\cdot$ in both arguments simultaneously. One can as well state it in each argument separately, by $\sum(aU) = a(\sum U)$ and $\sum(Ub) = (\sum U)b$ for all $a, b \in M, U \in \mathcal{A}M$, or combined to $a(\sum U)b = \sum aUb$.

Proposition 1. *Let M be a partially ordered monoid and $U, V \in \mathcal{P}M$ such that least upper bounds $u := \sum U$ and $v := \sum V$ exist. Then (i) implies (ii), where*

(i) for all $a, b \in M$, $\sum aUb = a(\sum U)b$ and $\sum aVb = a(\sum V)b$,
(ii) $\sum(UV) = \sum U \cdot \sum V$.

Notice that the existence of $\sum aUb$ and $\sum aVb$ in (i) and of $\sum(UV)$ in (ii) is not assumed, but is part of the claims.

Proof. Clearly, $UV < uv$. To prove that uv is $\sum(UV)$, we take $c \in M$ with $UV < c$ and show $uv \leq c$. For each $a \in U$, by (i), $\sum aV1$ exists, and as $aV1 \subseteq UV < c$, using (i) we have

$$av = a(\sum V)1 = \sum aV1 \leq c.$$

Hence $Uv = 1Uv < c$. By (i), $\sum 1Uv$ exists, and $uv = 1(\sum U)v = \sum 1Uv \leq c$.

Corollary 1. *If the partially ordered monoid M is $\mathcal{A}$-complete, the following conditions are equivalent:*

D_1) *(weak $\mathcal{A}$-distributivity): for all $a, b \in M$ and $U \in \mathcal{A}M$, $\sum aUb = a(\sum U)b$.*
D_2) *(strong $\mathcal{A}$-distributivity): for all $U, V \in \mathcal{A}M$, $\sum(UV) = \sum U \cdot \sum V$.*

An $\mathcal{A}$-*dioid* D is a partially ordered monoid that is $\mathcal{A}$-complete and $\mathcal{A}$-distributive.

Every $\mathcal{A}$-dioid $(M, \cdot, 1, \leq)$ is a dioid, using 0 and + defined by $0 := \sum \emptyset$ and $a + b := \sum\{a, b\}$. The zero and distributivity laws follow from D_1.

The monadic operator $\mathcal{A}$ provides us with a notion of continuous homomorphisms between $\mathcal{A}$-dioids, as follows.

D_3) A homomorphism $f : M \to M'$ is $\mathcal{A}$-*continuous*, if for all $U \in \mathcal{A}M$ and $y > \widetilde{f}(U)$ there is some $x > U$ with $y \geq f(x)$.

An $\mathcal{A}$-*morphism* is a $\leq$-preserving, $\mathcal{A}$-continuous monoid-homomorphism. We write $\mathbb{D}\mathcal{A}$ for the category of $\mathcal{A}$-dioids and $\mathcal{A}$-morphisms.

For order-preserving homomorphism $f : M \to M'$, $\mathcal{A}$-continuity of f reduces to:

$$f(\sum U) = \sum \widetilde{f}(U) \qquad \text{for all } U \in \mathcal{A}M.$$

Every $\mathcal{A}$-morphism is also a dioid-homomorphism.

Proposition 2. *If M is an $\mathcal{A}$-dioid, then for all $a, b \in M$ and $U, V \in \mathcal{A}M$:*

1. *$a(\sum U)b = \sum aUb$ and $\sum(UV) = \sum U \cdot \sum V$,*
2. *$a + (\sum U) = \sum(a + U)$ and $\sum(U + V) = \sum U + \sum V$.*

Proof. 2. Since $\{U, V\} \in \mathcal{F}\mathcal{A}M \subseteq \mathcal{A}\mathcal{A}M$, we have $U + V = \bigcup\{U, V\} \in \mathcal{A}M$, and so there is a least upper bound $\sum(U + V) \in M$. Hence

$$\sum U + \sum V \leq \sum(U + V) + \sum(U + V) = \sum(U + V).$$

Besides, $U + V < \sum U + \sum V$, so $\sum(U + V) \leq \sum U + \sum V$. Claim 1 follows from Corollary 1. □

The *free monoid* X^* generated by the set X consists of all finite sequences of elements from X, with concatenation as $\cdot$ and the empty sequence as 1. A *monomial* $m(x_1, \ldots, x_n)$ *in* $x_1, \ldots, x_n$ is a formal product of elements of X, and a *polynomial* $p(x_1, \ldots, x_n)$ a formal sum of monomials in $x_1, \ldots, x_n$. For elements $a_1, \ldots, a_n \in M$, we write $m^M(a_1, \ldots, a_n)$ for the value of the monomial in the monoid M and $p^M(a_1, \ldots, a_n)$ for the value of the polynomial p in the dioid M.

Corollary 2. *If M is an $\mathcal{A}$-dioid and $p(x_1, \ldots, x_n)$ a polynomial in $x_1, \ldots, x_n$, then $p^{\mathcal{A}M}(U_1, \ldots, U_n) \in \mathcal{A}M$ for all $U_1, \ldots, U_n \in \mathcal{A}M$, and*

$$\sum p^{\mathcal{A}M}(U_1, \ldots, U_n) = p^M(\sum U_1, \ldots, \sum U_n).$$

Proof. In an $\mathcal{A}$-dioid M, for $U, V \in \mathcal{A}M$ and $a, b \in M$ we have

$$(\sum U)(\sum V) = \sum(UV) \quad \text{and} \quad a(\sum U)b = \sum(aUb).$$

As $\mathcal{A}M \supseteq \mathcal{F}M$ is closed under product, we have $m^{\mathcal{A}M}(U_1, \dots, U_n) \in \mathcal{A}M$ for each monomial $m(x_1, \dots, x_n)$, and

$$m^M(\sum U_1, \dots, \sum U_n) = \sum m^{\mathcal{A}M}(U_1, \dots, U_n).$$

Since $\mathcal{A}M$ is closed under $\cup$ and $(\sum U) + (\sum V) = \sum(U \cup V)$, this extends to

$$p^M(\sum U_1, \dots, \sum U_n) = \sum p^{\mathcal{A}M}(U_1, \dots, U_n). \qquad \square$$

A polynomial in $x_1, \dots, x_n$ *over* M or *with parameters from* M is a polynomial in $x_1, \dots, x_n$ whose monomials may have additional factors taken from M. The corollary holds for polynomials with parameters as well.

3 $\mathcal{C}$-Dioids

Every monoid M gives rise to the idempotent semiring $(\mathcal{P}M, +, \cdot, \emptyset, \{1^M\})$. $\mathcal{P}M$ is *complete*: every $Y \subseteq \mathcal{P}M$ has a supremum, $\sum Y := \bigcup Y$, and the operations $+$ and $\cdot$ are $\sum$-*continuous*, i.e. for $Y, Z \subseteq \mathcal{P}M$ we have

$$\bigcup Y + \bigcup Z = \bigcup\{\, A + B \mid A \in Y, B \in Z \,\},$$
$$\bigcup Y \cdot \bigcup Z = \bigcup\{\, A \cdot B \mid A \in Y, B \in Z \,\}.$$

Any finite system $\boldsymbol{p}$ of polynomial inequations (with parameters from $\mathcal{P}M$)

$$x_1 \geq p_1(x_1, \dots, x_n), \ \dots, \ x_n \geq p_n(x_1, \dots, x_n)$$

has a least solution, a least $\boldsymbol{A} \in (\mathcal{P}M)^n$ with $\boldsymbol{A} \geq \boldsymbol{p}^{\mathcal{P}M}(\boldsymbol{A})$ (componentwise), where

$$\boldsymbol{A} := \bigcup\{\, \boldsymbol{A}_k \mid k \in \mathbb{N} \,\} \quad \text{with } A_i = \bigcup\{\, A_{i,k} \mid k \in \mathbb{N} \,\}$$

and the sequence of tuples $\boldsymbol{A}_k \in (\mathcal{P}M)^n$, $k \in \mathbb{N}$ is defined by

$$\boldsymbol{A}_0 := \emptyset^n, \qquad \boldsymbol{A}_{k+1} := \boldsymbol{p}^{\mathcal{P}M}(\boldsymbol{A}_k) := (p_1^{\mathcal{P}M}(\boldsymbol{A}_k), \dots, p_n^{\mathcal{P}M}(\boldsymbol{A}_k)).$$

For complete partial order $(P, \leq, 0)$ and $\sum$-continuous monotone $f : P \to P$ let μf be the least $a \in P$ with $f(a) \leq a$. A well-known fact about least solutions is:

Lemma 1. *Let $f : P \to P$ and $g : Q \to Q$ be $\sum$-continuous between the complete partial orders $(P, \leq^P, 0^P)$ and $(Q, \leq^Q, 0^Q)$. If $h : P \to Q$ is $\sum$-preserving with $h(0) = 0$ and $h \circ f = g \circ h$, then $h(\mu f) = \mu g$.*

We are henceforth considering idempotent semirings, like the $\mathcal{A}M$, which need not be complete: the existence of least upper bounds is restricted to $\mathcal{A}$-subsets.

For a monoid $M = (M, \cdot^M, 1^M)$, let $\mathcal{C}M$, the set of *context-free subsets of* M, be the closure of $\mathcal{F}M$ under (binary) union and under components of least solutions in $\mathcal{P}M$ of polynomial systems over $\mathcal{C}M$: the components $A_1, \ldots, A_n$ of the least solution in $\mathcal{P}M$ of any inequation system

$$x_1 \geq p_1(x_1, \ldots, x_n),\ \ldots,\ x_n \geq p_n(x_1, \ldots, x_n)$$

with polynomials p_i in $x_1, \ldots, x_n$ with parameters from $\mathcal{C}M$ belong to $\mathcal{C}M$.

This inductive definition of $\mathcal{C}M$ and $\mathcal{C}$ differs from the grammatical one in Hopkins [6]. We prove the equivalence of both definitions in Remark 2 below.

Example 3. For a monoid M with $a, b, c \in M$, the least solution of $x \geq \{a\} \cdot x \cdot \{b\} + \{c\}$ in $\mathcal{P}M$ is $L = \{\, a^n c b^n \mid n \in \mathbb{N} \,\}$, hence $L \in \mathcal{C}M$.

Theorem 1. *$\mathcal{C}$ is a monadic operator.*

Proof. Let M be a monoid M. By definition, $\mathcal{C}M$ satisfies A_0, A_1, as $\mathcal{F}M \subseteq \mathcal{C}M \subseteq \mathcal{P}M$, and it satisfies A_2 as it contains with $A, B \in \mathcal{C}M$ the least solution in $\mathcal{P}M$ of $x \geq Ay, y \geq B$, which is AB. By Theorem 9 of Hopkins [6], the conjunction of A_3 and A_4 is equivalent (over A_0, A_1, A_2) to the condition

A_5 For monoids M, N, every "substitution" homomorphism $\sigma : M \to \mathcal{C}N$ extends to a homomorphism $\sigma^* : \mathcal{C}M \to \mathcal{C}N$ by

$$\sigma^*(U) := \bigcup \{\, \sigma(m) \mid m \in U \,\}.$$

To show A_5, we prove $\sigma^*(U) \in \mathcal{C}N$ by induction on $U \in \mathcal{C}M$. If $U \in \mathcal{F}M$, then $\widetilde{\sigma}(U) \in \mathcal{F}\mathcal{C}N$, hence $\sigma^*(U) = \bigcup \widetilde{\sigma}(U) \in \mathcal{C}N$ since $\mathcal{C}N$ is closed under binary union. If $U = A \cup B$ with $A, B \in \mathcal{C}M$, then $\sigma^*(U) = \sigma^*(A) \cup \sigma^*(B) \in \mathcal{C}N$, by induction and closure of $\mathcal{C}N$ under binary union.

Finally, let U be the first component of the least solution in $\mathcal{P}M$ of the system $\boldsymbol{x} \geq \boldsymbol{p}(\boldsymbol{x})$ over $\mathcal{C}M$, where $\boldsymbol{p}(\boldsymbol{x}) = \boldsymbol{q}(\boldsymbol{x}, \boldsymbol{y})[\boldsymbol{y}/\boldsymbol{A}]$ and $\boldsymbol{A}$ are the parameters $A_j \in \mathcal{C}M$ in $\boldsymbol{p}(\boldsymbol{x})$. By induction, $\sigma^*(\boldsymbol{A}) \in \mathcal{C}N$. Let $\boldsymbol{x} \geq \boldsymbol{p}^{\sigma^*}(\boldsymbol{x})$ be the system over $\mathcal{C}N$, where $\boldsymbol{p}^{\sigma^*}(\boldsymbol{x}) := \boldsymbol{q}(\boldsymbol{x}, \boldsymbol{y})[\boldsymbol{y}/\sigma^*(\boldsymbol{A})]$ is obtained by replacing the parameters as shown. As $\sigma^* : \mathcal{P}M \to \mathcal{P}N$ is a homomorphism and preserves $\cup$, we have

$$\sigma^*(p_i^{\mathcal{P}M}(\boldsymbol{B})) = \sigma^*(q_i^{\mathcal{P}M}(\boldsymbol{B}, \boldsymbol{A})) = (q_i^{\mathcal{P}N}(\sigma^*(\boldsymbol{B}), \sigma^*(\boldsymbol{A})) = (p_i^{\sigma^*})^{\mathcal{P}N}(\sigma^*(\boldsymbol{B}))$$

for each i, so $\sigma^* \circ p_i = p_i^{\sigma^*} \circ \sigma^* : \mathcal{P}M \to \mathcal{P}N$. Since σ^* preserves $\bigcup$, it maps the least fixed-point of $\boldsymbol{p}$ in $(\mathcal{P}M)^n$ to the least fixed-point of $\boldsymbol{p}^{\sigma^*}$ in $(\mathcal{P}N)^n$, by Lemma 1. It follows that $\sigma^*(U)$ is the first component of the least solution of $\boldsymbol{p}^{\sigma^*}(\boldsymbol{x})$ in $\mathcal{P}N$, hence lies in $\mathcal{C}N$. □

Remark 2. Hopkins [6, 7] defines $\mathcal{C}$ differently, as follows. First, for free monoids X^*, one puts

$$\mathcal{C}X^* := \{\, L(G) \mid G \text{ is a context-free grammar over } X \,\}.$$

Here, a context-free grammar $G = (Q, S, H)$ over X has a set Q disjoint from X, with distinguished element $S \in Q$, and a finite set $H \subseteq Q \times (Q \cup X)^*$ of "context-free" rules. The language $L(G) \subseteq X^*$ is defined as usual. The class of context-free languages is closed under homomorphisms: If $h : X^* \to Y^*$ is a homomorphism, so is $\widetilde{h} : \mathcal{C}X^* \to \mathcal{C}Y^*$: a context-free grammar $G = (Q, S, H)$ over X gives rise to a context-free grammar $G^h = (Q, S, H^h)$ over Y with $\widetilde{h}(L(G)) = L(G^h)$.

For arbitrary monoid M, one takes a generating subset $X \subseteq M$ and uses the canonical homomorphism $h_X : X^* \to M$, where $h_X(x) = x$ for $x \in X$, to put

$$\mathcal{C}M = \{\, \widetilde{h_X}(L(G)) \mid G \text{ a context-free grammar over } X \,\}.$$

This is independent of the choice of X, as for any generating set $Y \subseteq M$ there is a homomorphism $h : X^* \to Y^*$ with $h_X = h_Y \circ h$, so that $\widetilde{h_X}(L(G)) = \widetilde{h_Y}(L(G^h))$.

Let $\mathcal{C}_H$ be the operator defined by Hopkins. To show $\mathcal{C} = \mathcal{C}_H$, fix a monoid M and a generating subset $Y \subseteq M$.

Claim. $\mathcal{C}_H M \subseteq \mathcal{C}M$.

Proof. Let $U \in \mathcal{C}_H M$ and $G = (X, S, H)$ a context-free grammar over Y, with finite set $H \subseteq X \times (X \cup Y)^*$, such that $U = \widetilde{h_Y}(L(G))$. For each nonterminal $x_i \in X = \{x_1, \dots, x_n\}$, let $p_i(\boldsymbol{x}, \boldsymbol{y}) = \sum\{\, m(\boldsymbol{x}, \boldsymbol{y}) \mid (x_i, m(\boldsymbol{x}, \boldsymbol{y})) \in H \,\}$. Then $\boldsymbol{x} \geq \boldsymbol{p}(\boldsymbol{x}, \boldsymbol{y})$ is a polynomial system over Y^*. Its least solution $\boldsymbol{A}$ in $\mathcal{P}Y^*$ consists of the languages $A_i = L_G(x_i) \subseteq Y^*$, where $L(G) = L_G(x_1)$, say. To show $\widetilde{h_Y}(\boldsymbol{A}) = (\mu \boldsymbol{x}\boldsymbol{p})^{\mathcal{P}M}(h_Y(\boldsymbol{y}))$ it is suffices by Lemma 1 that for any $\boldsymbol{B} \in \mathcal{P}Y^*$,

$$\widetilde{h_Y}(\boldsymbol{p}^{\mathcal{P}Y^*}(\boldsymbol{B}, \boldsymbol{y})) = \boldsymbol{p}^{\mathcal{P}M}(\widetilde{h_Y}(\boldsymbol{B}), h_Y(\boldsymbol{y})).$$

(Parameters y resp. $h_Y(y)$ are interpreted by the corresponding singleton sets.) But this is clear by the definition of h_Y. It follows that $U = \widetilde{h_Y}(L_G(x_1))$ is the first component of $\mu \boldsymbol{x}\boldsymbol{p}^{\mathcal{P}M}(h_Y(\boldsymbol{y}))$, and hence belongs to $\mathcal{C}M$. ◁

Claim. $\mathcal{C}M \subseteq \mathcal{C}_H M$.

Proof. We proceed by induction on $U \in \mathcal{C}M$. If $U = \{w_1, \dots, w_m\} \in \mathcal{F}M$, there are $y_1, \dots, y_m \in Y^*$ with $h_Y(y_1) = w_1, \dots, h_y(y_m) = w_m$ and a context-free grammar $G = (X, S, H)$ over Y with $L(G) = \{y_1, \dots, y_m\}$. Hence $U = \{h_Y(y_1), \dots, h_Y(y_m)\} = \widetilde{h_Y}(L(G)) \in \mathcal{C}_H M$. The case $U = U_1 \cup U_2$ is left to the reader. Finally, let U be a component of the least solution $\boldsymbol{U}$ in $\mathcal{P}M$ of a polynomial system $\boldsymbol{x} \geq \boldsymbol{p}(\boldsymbol{x})$ over $\mathcal{C}M$, using $\boldsymbol{p}(\boldsymbol{x}) = \boldsymbol{q}(\boldsymbol{x}, \boldsymbol{B})$ to show the parameters $\boldsymbol{B}$ from $\mathcal{C}M$. By induction, for each B_j of $\boldsymbol{B}$ there is a context-free grammar $G_j = (X_j, S_j, H_j)$ over Y such that $B_j = \widetilde{h_Y}(L(G_j))$. Let X be the disjoint union of $\boldsymbol{x}$ with these X_j. Each inequation $x_i \geq q_i(\boldsymbol{x}, \boldsymbol{B})$ of $\boldsymbol{x} \geq \boldsymbol{p}(\boldsymbol{x})$ gives rise to context-free grammar rules $(x_i, m_{i,1}(\boldsymbol{x}, \boldsymbol{S})), \dots, (x_i, m_{i,n_i}(\boldsymbol{x}, \boldsymbol{S})) \in X \times (X \cup Y)^*$, where $m_{i,k}(\boldsymbol{x}, \boldsymbol{B})$ are the monomials of $q_i(\boldsymbol{x}, \boldsymbol{B})$. Let $G(X, S, H)$ be the context-free grammar over Y where S is the variable from $\boldsymbol{x}$ corresponding to U and H is the union of the H_j with the rules obtained from the inequations $x_i \geq q_i(\boldsymbol{x}, \boldsymbol{S})$, $i = 1, \dots, n$. Then $L_G(S_j) = L(G_j)$, hence $\widetilde{h_Y}(L_G(S_j)) = \widetilde{h_Y}(L(G_j)) = B_j$, and $\widetilde{h_Y}(L_G(x_i)) = U_i$. Therefore, $U = \widetilde{h_Y}(L(G))$. ◁

4 μ-Continuous Chomsky Algebras

Let X be an infinite set of variables. The set of *μ-terms over X* is defined by the grammar

$$t := x \mid 0 \mid 1 \mid (s \cdot t) \mid (s + t) \mid \mu x t.$$

A term not containing μ will be called (somewhat unprecisely) a *polynomial*. The free occurrences of variables in a term are defined as usual. By *free*(t) we denote the set of variables having a free occurrence in t; in particular, *free*$(\mu x t) =$ *free*$(t) \setminus \{x\}$. By $t(x_1, \ldots, x_n)$ we indicate *free*$(t) \subseteq \{x_1, \ldots, x_n\}$. In $\mu x t$ all free occurrences of x in t are *bound* by μx. By $t[x/s]$ we denote the result of substituting all free occurrences of x in t by s, renaming bound variables of t to avoid capture of free variables of s by bindings in t.

A *partially ordered μ-semiring* $(M, +, \cdot, 0, 1, \leq)$ is a semiring $(M, +, \cdot, 0, 1)$ with a partial order $\leq$ on M, where every term t defines a function $t^M : (X \to M) \to M$, so that for all variables $x \in X$ and terms s, t we have:

1. for all valuations $g : X \to M$,

$$\begin{array}{lrcl} 0^M(g) = 0, & (s+t)^M(g) & = & s^M(g) + t^M(g), \\ 1^M(g) = 1, & (s \cdot t)^M(g) & = & s^M(g) \cdot t^M(g), \\ x^M(g) = g(x), & \multicolumn{3}{l}{\text{if } s^M \leq t^M, \text{ then } \mu x s^M \leq \mu x t^M,} \end{array}$$

2. t^M is monotone with respect to the pointwise order on $X \to M$,
3. $t^M(g) = t^M(h)$, for all valuations $g, h : X \to M$ which agree on *free*(t),
4. $t[x/s]^M(g) = t^M(g[x/s^M(g)])$, for all valuations $g : X \to M$.

When *free*$(t) \subseteq \{x_1, \ldots, x_n\}$ and $g(x_i) = a_i$ for $1 \leq i \leq n$, instead of $t^M(g)$ we often write $t^M[x_1/a_1, \ldots, x_n/a_n]$ or just $t^M(a_1, \ldots, a_n)$.

The final two conditions above are called the *coincidence* and *substitution properties*; in the latter, $g[x/a]$ denotes the valuation that agrees with g, except that it assigns a to x. Clearly, the substitution property extends to simultaneous substitutions $[x_1/s_1, \ldots, x_n/s_n]$.

Following Grathwohl et al. [3], an idempotent semiring $(M, +, \cdot, 0, 1)$ is *algebraically closed* or a *Chomsky algebra*, if every finite system of polynomial inequations

$$\begin{array}{ll} x_1 \geq p_1(x_1, \ldots, x_n, y_1, \ldots, y_m), & \\ \quad\vdots & \\ x_n \geq p_n(x_1, \ldots, x_n, y_1, \ldots, y_m), & \text{abbreviated } \boldsymbol{x} \geq \boldsymbol{p}(\boldsymbol{x}, \boldsymbol{y}), \end{array} \tag{1}$$

has least solutions, i.e. for all $\boldsymbol{b} \in M^m$ there is a least $\boldsymbol{a} = a_1, \ldots, a_n \in M^n$ such that $a_i \geq p_i^M(\boldsymbol{a}, \boldsymbol{b})$ for $i = 1, \ldots, n$, where $\leq$ is the natural partial order on M defined by $a \leq b$ iff $a + b = b$. Of course, for each $\boldsymbol{b}$ the least solution $\boldsymbol{a}$ is unique.

Example 4. The set $\mathcal{C}X^*$ of *context-free languages* over X is the smallest set $\mathcal{L} \subseteq \mathcal{P}X^*$ such that (i) each finite subset of $X \cup \{\epsilon\}$ is in $\mathcal{L}$ and (ii) if $\boldsymbol{x} \geq \boldsymbol{p}(\boldsymbol{x}, \boldsymbol{y})$ is a polynomial system, and $\boldsymbol{B} = B_1, \ldots, B_m \in \mathcal{L}$, then the components A_i of the least $\boldsymbol{A} = A_1, \ldots, A_n \in \mathcal{P}X^*$ with $\boldsymbol{A} \supseteq \boldsymbol{p}^{\mathcal{P}X^*}(\boldsymbol{A}, \boldsymbol{B})$ belong to $\mathcal{L}$. With the operations inherited from $\mathcal{P}X^*$, $(\mathcal{C}X^*, +, \cdot, 0, 1)$ is a Chomsky algebra. For example, $\{\, a^n b^n \mid n \in \mathbb{N} \,\}$ is a context-free language over $X \supseteq \{a, b\}$; it is the least solution of $x \geq axb + 1$ relative to the valuation $g(a) = \{a\}, g(b) = \{b\}$.

The class of regular languages over X does not form a Chomsky algebra.

Lemma 2. *(Grathwohl et al. [3]) Every Chomsky algebra M is an idempotent, partially ordered μ-semiring, if for all μ-terms t, variables x and valuations $g : X \to M$ we take*

$$\mu x t^M(g) := \text{the least } a \in M \text{ such that } t^M(g[x/a]) \leq a. \tag{2}$$

Moreover, every inequation system $\mathbf{t}(\boldsymbol{x}, \boldsymbol{y}) \leq \boldsymbol{x}$ with μ-terms $\mathbf{t}(\boldsymbol{x}, \boldsymbol{y})$ has least solutions in M, i.e. for all parameters $\boldsymbol{b}$ from M there is a least tuple $\boldsymbol{a}$ in M such that $\mathbf{t}^M(\boldsymbol{a}, \boldsymbol{b}) \leq \boldsymbol{a}$.

Proof. See Lemma 2.1 in [3] or, in more detail, Lemma 8 in [11].

A Chomsky algebra M is *μ-continuous*, if for all $a, b \in M$, all μ-terms t, variables x and valuations $g : X \to M$ it satisfies the μ-continuity condition

$$a \cdot \mu x t^M(g) \cdot b = \sum \{\, a \cdot m x t^M(g) \cdot b \mid m \in \mathbb{N} \,\}, \tag{3}$$

where mxt is defined inductively by $0xt := 0$, $(m+1)xt := t[x/mxt]$. This condition of Grathwohl et al. [3] generalizes the *-continuity condition

$$a \cdot c^* \cdot b = \sum \{\, a \cdot c^m \cdot b \mid m \in \mathbb{N} \,\}$$

of Kozen [9]; every μ-continuous Chomsky algebra is a *-continuous Kleene algebra, if c^* is defined by $\mu x(cx + 1)$.

To prove that every μ-continuous Chomsky algebra is a $\mathcal{C}$-dioid, we will below need a vector-version of the μ-continuity condition. By a theorem of Bekić [1], deBakker and Scott [2], the n-ary least-fixed-point operator can be reduced to the unary one. The theorem applies to complete (or at least ω-complete) partial orders only, but $\mathcal{C}X^*$ is far from being ω-complete. So it needs to be checked that the suprema used in Bekić's reduction exist also in the incomplete partial orders of $\mathcal{C}M$. This can be done, leading to a definition of term vectors $\mu\boldsymbol{x}\boldsymbol{t}$ that embody Bekić's reduction (cf. Leiß and Ésik [12]).

For vectors $\boldsymbol{t} = t_1, \ldots, t_n$ of terms and $\boldsymbol{x} = x_1, \ldots, x_n$ of pairwise different variables, we define the term vector $\mu\boldsymbol{x}\boldsymbol{t}$ as follows. If $n = 1$, then $\mu\boldsymbol{x}\boldsymbol{t} := \mu x_1 t_1$. If $n > 1$, $\boldsymbol{x} = (\boldsymbol{y}, z)$ and $\boldsymbol{t} = (\boldsymbol{r}, s)$ with term vectors $\boldsymbol{r}, s$ of lengths $|\boldsymbol{y}|, |z| < n$, then $\mu\boldsymbol{x}\boldsymbol{t}$ is

$$\mu(\boldsymbol{y}, z)(\boldsymbol{r}, s) := (\mu\boldsymbol{y}.\boldsymbol{r}[z/\mu z s], \mu z.s[\boldsymbol{y}/\mu\boldsymbol{y}\boldsymbol{r}]). \tag{4}$$

It can be shown that the values of $\mu\boldsymbol{x}\boldsymbol{t}$ in Chomsky algebras do not depend on the choice of splitting $\boldsymbol{x}$ into $\boldsymbol{y}, z$ in the definition.

Lemma 3. *For any Chomsky algebra M and valuation $g : X \to M$, $\mu\boldsymbol{x}\boldsymbol{t}^M(g)$ is the least tuple $\boldsymbol{a}$ in M such that $\boldsymbol{t}^M(g[\boldsymbol{x}/\boldsymbol{a}]) \leq \boldsymbol{a}$.*

Proof. See Lemma 14 of Leiß [11].

It is essential that the unary version of μ-continuity implies the n-ary version:

Lemma 4. *Let M be a μ-continuous Chomsky algebra and $g : X \to M$. Then*

$$\boldsymbol{a} \cdot \mu \boldsymbol{x}\boldsymbol{t}^M(g) \cdot \boldsymbol{b} = \sum \{ \boldsymbol{a} \cdot m\boldsymbol{x}\boldsymbol{t}^M(g) \cdot \boldsymbol{b} \mid m \in \mathbb{N} \},$$

for any term vector $\boldsymbol{t}$ and vectors $\boldsymbol{a}, \boldsymbol{b}$ of elements of M of the same length as $\boldsymbol{t}$.

Proof. See Corollary 23 in Leiß [11].

5 $\mathcal{C}$-Dioids and μ-Continuous Chomsky Algebras

We now prove our result, that the categories of $\mathcal{C}$-dioids and μ-continuous Chomsky algebras are the same.

Theorem 2. *Every $\mathcal{C}$-dioid M is a μ-continuous Chomsky algebra.*

Proof. To simplify the notation, we talk of a "polynomial system $\boldsymbol{x} \geq \boldsymbol{p}(\boldsymbol{x})$ *over* M" or *with parameters from* M, meaning that $\boldsymbol{x} \geq \boldsymbol{p}(\boldsymbol{x}, \boldsymbol{y})$ is considered with a fixed valuation $g : X \to M$ that provides the values for the parameters $\boldsymbol{y}$. Hence we also write $\mu \boldsymbol{x}\boldsymbol{p}^M$ instead of $\mu \boldsymbol{x}\boldsymbol{p}^M(g)$.

(i) M is algebraically closed: Let $\boldsymbol{x} \geq \boldsymbol{p}(\boldsymbol{x})$ be a polynomial system with parameters $\boldsymbol{b}$ from M. In $\mathcal{P}M$ it has a least solution $\boldsymbol{A} = \mu \boldsymbol{x}\boldsymbol{p}^{\mathcal{C}M}$ with components $A_i \in \mathcal{C}M$ (where parameters $b \in M$ are interpreted by $\{b\} \in \mathcal{F}M \subseteq \mathcal{C}M$). Since M is a $\mathcal{C}$-dioid, the suprema $a_i := \sum A_i \in M$ $(1 \leq i \leq n)$ exist. We show that $\boldsymbol{a} = (a_1, \dots, a_n) = \sum \boldsymbol{A}$ is the least solution of $\boldsymbol{x} \geq \boldsymbol{p}(\boldsymbol{x})$ in M, i.e.

$$\mu \boldsymbol{x}\boldsymbol{p}^M = \sum \mu \boldsymbol{x}\boldsymbol{p}^{\mathcal{C}M} = \sum \boldsymbol{A}. \tag{5}$$

By the distributivity properties of $\mathcal{C}$-dioids, $\boldsymbol{a}$ is a solution of $\boldsymbol{x} \geq \boldsymbol{p}(\boldsymbol{x})$ in M, using Corollary 2:

$$p_i^M(\boldsymbol{a}) = p_i^M(\sum A_1, \dots, \sum A_n) = \sum p_i^{\mathcal{C}M}(A_1, \dots, A_n) \ \leq \ \sum A_i = a_i.$$

To show that $\boldsymbol{a}$ is the least solution of $\boldsymbol{x} \geq \boldsymbol{p}(\boldsymbol{x})$ in M, let $\boldsymbol{c}$ be any solution in M, hence $c_i \geq p_i^M(\boldsymbol{c})$ for $1 \leq i \leq n$. It is sufficient to show $\boldsymbol{c} > \boldsymbol{A}$, because $\boldsymbol{a}$ is the least upper bound of $\boldsymbol{A}$. We know that

$$A_i = \bigcup \{ p_i^{\mathcal{P}M}(\boldsymbol{A}_m) \mid m \in \mathbb{N} \}$$

where $\boldsymbol{A}_0 := \emptyset$, $\boldsymbol{A}_{m+1} := \boldsymbol{p}^{\mathcal{P}M}(\boldsymbol{A}_m)$. For $m = 0$, obviously $\boldsymbol{c} > \boldsymbol{A}_0$. Suppose $\boldsymbol{c} > \boldsymbol{A}_m$ for some m. By induction on p_i, $p_i^{\mathcal{C}M}(\boldsymbol{A}_m) < p_i^M(\boldsymbol{c})$ for each i, hence $\boldsymbol{A}_{m+1} < \boldsymbol{p}^M(\boldsymbol{c}) \leq \boldsymbol{c}$. Therefore, $\boldsymbol{A} < \boldsymbol{c}$.

(ii) M is μ-continuous: Any valuation $g : X \to M$ is the composition of a valuation $g' : X \to \mathcal{C}M$ with $\sum : \mathcal{C}M \to M$: if $g'(x) = \{g(x)\}$, $g(x) = \sum g'(x)$.

Claim. For every μ-term $t(x_1,\dots,x_n)$ and sets $A_1,\dots,A_n \in \mathcal{C}M$,

$$t^M(\sum A_1,\dots,\sum A_n) = \sum t^{\mathcal{C}M}(A_1,\dots,A_n). \tag{6}$$

Proof. By induction on t. We abbreviate $A_1,\dots,A_n$ by $\boldsymbol{A}$. For atomic terms 0 and 1, $0^M = \sum\emptyset = \sum 0^{\mathcal{C}M}$ and $1^M = \sum\{1^M\} = \sum 1^{\mathcal{C}M}$. For variables,

$$x_i^M(\sum \boldsymbol{A}) = \sum A_i = \sum x_i^{\mathcal{C}M}(\boldsymbol{A}).$$

For term $(r+s)$,

$$\begin{aligned}(r+s)^M(\sum \boldsymbol{A}) &= r^M(\sum \boldsymbol{A}) +^M s^M(\sum \boldsymbol{A})\\ &= \sum\{\sum r^{\mathcal{C}M}(\boldsymbol{A}), \sum s^{\mathcal{C}M}(\boldsymbol{A})\}\\ &= \sum(r^{\mathcal{C}M}(\boldsymbol{A}) \cup s^{\mathcal{C}M}(\boldsymbol{A}))\\ &= \sum(r+s)^{\mathcal{C}M}(\boldsymbol{A}).\end{aligned}$$

For term $(r\cdot s)$, we use the distributivity property of the $\mathcal{C}$-dioid M

$$\begin{aligned}(r\cdot s)^M(\sum \boldsymbol{A}) &= r^M(\sum \boldsymbol{A}) \cdot^M s^M(\sum \boldsymbol{A})\\ &= (\sum r^{\mathcal{C}M}(\boldsymbol{A})) \cdot^M (\sum s^{\mathcal{C}M}(\boldsymbol{A}))\\ &= \sum(r^{\mathcal{C}M}(\boldsymbol{A}) \cdot^{\mathcal{C}M} s^{\mathcal{C}M}(\boldsymbol{A}))\\ &= \sum(r\cdot s)^{\mathcal{C}M}(\boldsymbol{A}).\end{aligned}$$

Finally, consider μxr. Let $f:\mathcal{C}M\to\mathcal{C}M$ be $f(B) = r^{\mathcal{C}M}(\boldsymbol{A},B)$, $g: M\to M$ be $g(b) = r^M(\sum\boldsymbol{A},b)$. By the induction hypothesis, we have $h\circ f = g\circ h$ for $h=\sum$, i.e. for each $B\in\mathcal{C}M$,

$$\sum r^{\mathcal{C}M}(\boldsymbol{A},B) = r^M(\sum\boldsymbol{A},\sum B).$$

We would like to use Lemma 1 to conclude that $\sum = h$ maps the least fixed-point of f, $\mu xr^{\mathcal{C}M}(\boldsymbol{A})$, to the least fixed point of g, $\mu xr^M(\sum\boldsymbol{A})$. The lemma does not apply literally, since the orders on $\mathcal{C}M$ and M are *not* complete. For $\mathcal{C}M$, this is not a problem, since we *do* have $\mu xr^{\mathcal{C}M}(\boldsymbol{A}) = \bigcup\{\, mxr^{\mathcal{C}M}(\boldsymbol{A}) \mid m\in\mathbb{N}\,\}$, because $t^{\mathcal{C}M}(v) = t^{\mathcal{P}M}(v)$ for each valuation $v: X\to\mathcal{C}M$ (c.f. Lemma 10 in [11]), and the equation holds in $\mathcal{P}M$, which *is* complete. For M, by the argument below the least fixed point of g also is the $\sum$ of the finite iterations of g, i.e.

$$\mu xr^M(\sum\boldsymbol{A}) = \sum\{\, mxr^M(\sum\boldsymbol{A}) \mid m\in\mathbb{N}\,\}.$$

Namely, by induction along the well-ordering $\prec$ on μ-terms of Kozen [10], in which $mxr \prec \mu xr$ for each $m\in\mathbb{N}$, we may also assume $mxr^M(\sum\boldsymbol{A}) =$

$\sum mxr^{\mathcal{C}M}(\boldsymbol{A})$ for all $m \in \mathbb{N}$. Therefore,

$$\begin{aligned}\sum \mu xr^{\mathcal{C}M}(\boldsymbol{A}) &= \sum \bigcup \{\, mxr^{\mathcal{C}M}(\boldsymbol{A}) \mid m \in \mathbb{N}\,\} \\ &= \sum \{\, \sum mxr^{\mathcal{C}M}(\boldsymbol{A}) \mid m \in \mathbb{N}\,\} \\ &= \sum \{\, mxr^{M}(\sum \boldsymbol{A}) \mid m \in \mathbb{N}\,\}.\end{aligned}$$

This implies $\sum \mu xr^{\mathcal{C}M}(\boldsymbol{A}) \leq \mu xr^{M}(\sum \boldsymbol{A})$, since $mxr^{M}(\sum \boldsymbol{A}) \leq \mu xr^{M}(\sum \boldsymbol{A})$ by induction on m. On the other hand, $\sum \mu xr^{\mathcal{C}M}(\boldsymbol{A})$ solves $r^{M}(\sum \boldsymbol{A}, x) \leq x$ in M, by the induction hypothesis for r and a fixed-point property of $\mu xr^{\mathcal{C}M}(\boldsymbol{A})$:

$$r^{M}(\sum \boldsymbol{A}, \sum \mu xr^{\mathcal{C}M}(\boldsymbol{A})) = \sum r^{\mathcal{C}M}(\boldsymbol{A}, \mu xr^{\mathcal{C}M}(\boldsymbol{A})) \leq \sum \mu xr^{\mathcal{C}M}(\boldsymbol{A}).$$

Hence, $\sum \mu xr^{\mathcal{C}M}(\boldsymbol{A})$ lies above the least solution of $r^{M}(\sum \boldsymbol{A}, x) \leq x$, i.e. we have the reverse inequation $\mu xr^{M}(\sum \boldsymbol{A}) \leq \sum \mu xr^{\mathcal{C}M}(\boldsymbol{A})$ also. $\triangleleft$

We can now prove the μ-continuity condition (3) for valuations $g : X \to M$ of the form $g = \sum g'$ for some $g' : X \to \mathcal{C}M$.

Claim. For all μ-terms μxt with $t(x, \boldsymbol{x})$, all $\boldsymbol{A} = A_1, \ldots, A_n \in \mathcal{C}M$, and $a, b \in M$:

$$a \cdot \mu xt^{M}(\sum \boldsymbol{A}) \cdot b = \sum \{\, a \cdot mxt^{M}(\sum \boldsymbol{A}) \cdot b \mid m \in \mathbb{N}\,\}. \tag{7}$$

Proof. Using the previous Claim (6) in the first and last step, we have

$$\begin{aligned} a \cdot \mu xt^{M}(\sum \boldsymbol{A}) \cdot b &= (\sum \{a\})(\sum \mu xt^{\mathcal{C}M}(\boldsymbol{A}))(\sum \{b\}) \\ &= \sum (\{a\} \cdot \mu xt^{\mathcal{C}M}(\boldsymbol{A}) \cdot \{b\}) \\ &= \sum (\{a\} \cdot \bigcup \{\, mxt^{\mathcal{C}M}(\boldsymbol{A}) \mid m \in \mathbb{N}\,\} \cdot \{b\}) \\ &= \sum (\bigcup \{\, \{a\} \cdot mxt^{\mathcal{C}M}(\boldsymbol{A}) \cdot \{b\} \mid m \in \mathbb{N}\,\}) \\ &= \sum \{\, \sum (\{a\} \cdot mxt^{\mathcal{C}M}(\boldsymbol{A}) \cdot \{b\}) \mid m \in \mathbb{N}\,\} \\ &= \sum \{\, (\sum \{a\}) \cdot (\sum mxt^{\mathcal{C}M}(\boldsymbol{A})) \cdot (\sum \{b\}) \mid m \in \mathbb{N}\,\} \\ &= \sum \{\, a \cdot mxt^{M}(\sum \boldsymbol{A}) \cdot b \mid m \in \mathbb{N}\,\}. \qquad \triangleleft \end{aligned}$$

This completes the proof of the theorem. $\square$

We now come to the reverse inclusion, that every μ-continuous Chomsky algebra is a $\mathcal{C}$-dioid, i.e. is $\mathcal{C}$-complete and $\mathcal{C}$-distributive.

The idea is, of course, that for any polynomial system $\boldsymbol{x} \geq \boldsymbol{p}(\boldsymbol{x}, \boldsymbol{y})$, if $U \in \mathcal{C}M$ is a component of the least solution $\boldsymbol{U}$ of $\boldsymbol{x} \geq \boldsymbol{p}^{\mathcal{C}M}(\boldsymbol{x}, \boldsymbol{A})$ with parameters $\boldsymbol{A}$, then all components of $\boldsymbol{U}$ have least upper bounds, namely the components of the least solution $\boldsymbol{u}$ of $\boldsymbol{x} \geq \boldsymbol{p}^{M}(\boldsymbol{x}, \sum \boldsymbol{A})$ in M. And since $\boldsymbol{U}$ is the union of finite iterations $\boldsymbol{U}_m = m\boldsymbol{x}\boldsymbol{p}^{\mathcal{C}M}(\boldsymbol{A})$, these $\boldsymbol{U}_m$ should have least upper bounds $\boldsymbol{u}_m = m\boldsymbol{x}\boldsymbol{p}^{M}(\sum \boldsymbol{A})$, which make up $\boldsymbol{u} = \sum \{\, \boldsymbol{u}_m \mid m \in \mathbb{N}\,\}$. Since the p_i contain products, to show $\boldsymbol{u}_m = \sum \boldsymbol{U}_m$ the induction must provide the distributivity property $\sum (UV) = (\sum U)(\sum V)$ for all U, V among $\boldsymbol{U}_m, \boldsymbol{A}$.

Theorem 3. *Every μ-continuous Chomsky algebra M is a $\mathcal{C}$-dioid.*

Proof. By induction on the construction of $\mathcal{C}M$, we show that for all $U, V \in \mathcal{C}M$

a) U and V have least upper bounds, $\sum U$ resp. $\sum V \in M$, and
b) UV has a least upper bound, and $\sum(UV) = (\sum U)(\sum V)$.

Then M is a $\mathcal{C}$-dioid. By Proposition 1, it is sufficient to show a) and

b') For all $a, b \in M$, $\sum(aUb) = a(\sum U)b$ and $\sum(aVb) = a(\sum V)b$.

If U, V belong to $\mathcal{F}M$, a) and b') are true since M is a dioid. Otherwise, there is a polynomial system $\boldsymbol{x} \geq \boldsymbol{p}(\boldsymbol{x}, \boldsymbol{y})$ and parameters $\boldsymbol{A} \in (\mathcal{C}M)^k$ such that U, V belong to the least solution $\boldsymbol{U}$ of $\boldsymbol{x} \geq \boldsymbol{p}^{\mathcal{P}M}(\boldsymbol{x}, \boldsymbol{A})$ in $\mathcal{P}M$. By induction, a) and b'), hence b), hold for all $U, V \in \boldsymbol{A}$. We must show a) and b') for all $U, V \in \boldsymbol{U}$.

By induction on m, we first prove for $\boldsymbol{U}_m := m\boldsymbol{x}\boldsymbol{p}^{\mathcal{C}M}(\boldsymbol{A})$, $\boldsymbol{u}_m := m\boldsymbol{x}\boldsymbol{p}^M(\sum \boldsymbol{A})$

(i) $\sum \boldsymbol{U}_m$ exists (componentwise),
(ii) for all monomials $q(\boldsymbol{x}, \boldsymbol{y})$, $q^M(\sum \boldsymbol{U}_m, \sum \boldsymbol{A}) = \sum q^{\mathcal{C}M}(\boldsymbol{U}_m, \boldsymbol{A})$,
(iii) $\boldsymbol{u}_m = \sum \boldsymbol{U}_m$.

Clearly, (ii) extends to polynomials $q(\boldsymbol{x}, \boldsymbol{y})$, as $\sum(A \cup B) = \sum A + \sum B$.

For $m = 0$, (i) and (iii) are clear: $\mathbf{0} = \sum \boldsymbol{\emptyset}$. Therefore, (ii) follows from the hypothesis a) and b) for members of $\boldsymbol{A}$.

For $m + 1$, by induction $\sum \boldsymbol{U}_m$ exists by (i), and then

$$
\begin{aligned}
\boldsymbol{u}_{m+1} &= \boldsymbol{p}^M(\boldsymbol{u}_m, \sum \boldsymbol{A}) && \text{(by definition)}\\
&= \boldsymbol{p}^M(\sum \boldsymbol{U}_m, \sum \boldsymbol{A}) && \text{(by (iii))}\\
&= \sum \boldsymbol{p}^{\mathcal{C}M}(\boldsymbol{U}_m, \boldsymbol{A}) && \text{(by (ii))}\\
&= \sum \boldsymbol{U}_{m+1}. && \text{(by definition)}
\end{aligned}
$$

Hence, (i) $\sum \boldsymbol{U}_{m+1}$ exists, and (iii) $\boldsymbol{u}_{m+1} = \sum \boldsymbol{U}_{m+1}$. For (ii), let $q(\boldsymbol{x}, \boldsymbol{y})$ be a monomial in $\boldsymbol{x}, \boldsymbol{y}$, and $r(\boldsymbol{x}, \boldsymbol{y})$ the polynomial obtained by distribution from $q(\boldsymbol{x}, \boldsymbol{y})[\boldsymbol{x}/\boldsymbol{p}(\boldsymbol{x}, \boldsymbol{y})]$. Then

$$
\begin{aligned}
q^M(\sum \boldsymbol{U}_{m+1}, \sum \boldsymbol{A}) &= r^M(\sum \boldsymbol{U}_m, \sum \boldsymbol{A})\\
&= \sum r^{\mathcal{C}M}(\boldsymbol{U}_m, \boldsymbol{A}) && \text{(by (ii) for } r)\\
&= \sum q^{\mathcal{C}M}(\boldsymbol{U}_{m+1}, \boldsymbol{A}).
\end{aligned}
$$

Since M is a Chomsky algebra, $\boldsymbol{x} \geq \boldsymbol{p}^M(\boldsymbol{x}, \sum \boldsymbol{A})$ has a least solution, $\boldsymbol{u} := \mu \boldsymbol{x}\boldsymbol{p}^M(\sum \boldsymbol{A})$. It follows that it is the least upper bound of $\boldsymbol{U} = \mu \boldsymbol{x}\boldsymbol{p}^{\mathcal{C}M}(\boldsymbol{A})$:

$$\begin{aligned}
\boldsymbol{u} &= \mu \boldsymbol{x}\boldsymbol{p}^M(\textstyle\sum \boldsymbol{A}) \\
&= \textstyle\sum\{\, m\boldsymbol{x}\boldsymbol{p}^M(\sum \boldsymbol{A}) \mid m \in \mathbb{N}\,\} && (M \text{ is } \mu\text{-continuous}) \\
&= \textstyle\sum\{\, \boldsymbol{u}_m \mid m \in \mathbb{N}\,\} \\
&= \textstyle\sum\{\, \sum \boldsymbol{U}_m \mid m \in \mathbb{N}\,\} && (\text{by (iii)}) \\
&= \textstyle\sum \bigcup\{\, \boldsymbol{U}_m \mid m \in \mathbb{N}\,\} \\
&= \textstyle\sum \boldsymbol{U} = \sum \mu \boldsymbol{x}\boldsymbol{p}^{\mathcal{C}M}(\boldsymbol{A}).
\end{aligned}$$

In particular, we have shown a) for $U, V \in \boldsymbol{U}$. To show b') extend a, b to some $\boldsymbol{a}, \boldsymbol{b} \in M^n$. Having $\boldsymbol{a}(\sum \boldsymbol{U}_m)\boldsymbol{b} = \sum \boldsymbol{a}\boldsymbol{U}_m\boldsymbol{b}$ inductively by (ii), we obtain

$$\begin{aligned}
\boldsymbol{a}(\textstyle\sum \boldsymbol{U})\boldsymbol{b} &= \boldsymbol{a} \cdot \boldsymbol{u} \cdot \boldsymbol{b} \\
&= \textstyle\sum\{\, \boldsymbol{a} \cdot \boldsymbol{u}_m \cdot \boldsymbol{b} \mid m \in \mathbb{N}\,\} && (M \text{ is } \mu\text{-continuous}) \\
&= \textstyle\sum\{\, \boldsymbol{a}(\sum \boldsymbol{U}_m)\boldsymbol{b} \mid m \in \mathbb{N}\,\} && (\text{by (iii)}) \\
&= \textstyle\sum\{\, \sum(\boldsymbol{a}U_m\boldsymbol{b}) \mid m \in \mathbb{N}\,\} && (\text{by (ii)}) \\
&= \textstyle\sum \bigcup\{\, \boldsymbol{a}\boldsymbol{U}_m\boldsymbol{b} \mid m \in \mathbb{N}\,\} && (\textstyle\sum \text{ property}) \\
&= \textstyle\sum(\boldsymbol{a} \cdot \bigcup\{\, \boldsymbol{U}_m \mid m \in \mathbb{N}\,\} \cdot \boldsymbol{b}) && (\cdot^{\mathcal{C}M} \text{ is } \textstyle\bigcup\text{-continuous}) \\
&= \textstyle\sum(\boldsymbol{a}\boldsymbol{U}\boldsymbol{b}).
\end{aligned}$$

Hence, for $U \in \boldsymbol{U}$ we have b') $a(\sum U)b = \sum aUb$ for all a, b. □

The morphisms in the category $\mathbb{D}\mathcal{A}$ of $\mathcal{C}$-dioids are the $\mathcal{C}$-morphisms. In the category of μ-continuous Chomsky algebras of Grathwohl et al. [3], the morphisms are the semiring homomorphisms that preserve least solutions of polynomial inequalities. It remains to be checked that these two types of morphisms are the same.

Proposition 3. *Let $f : M \to M'$ be a homomorphism between $\mathcal{C}$-dioids M and M'. Then f is a $\mathcal{C}$-morphism iff f is a semiring homomorphism that perserves least solutions of polynomial inequalities.*

Proof. If f is a dioid-homomorphism, we have $\widetilde{f}(m\boldsymbol{x}\boldsymbol{p}^{\mathcal{C}M}(\boldsymbol{A})) = m\boldsymbol{x}\boldsymbol{p}^{\mathcal{C}M'}(\widetilde{f}(\boldsymbol{A}))$ for all m, which implies $\widetilde{f}(\mu \boldsymbol{x}\boldsymbol{p}^{\mathcal{C}M}(\boldsymbol{A})) = \mu \boldsymbol{x}\boldsymbol{p}^{\mathcal{C}M'}(\widetilde{f}(\boldsymbol{A}))$.

⇒: Let f be a $\mathcal{C}$-morphism, $\boldsymbol{x} \geq \boldsymbol{p}(\boldsymbol{x}, \boldsymbol{y})$ a system of polynomial inequalities with $n = |\boldsymbol{x}|, k = |\boldsymbol{y}|$ and $\boldsymbol{a} \in M^k$. We have $\boldsymbol{a} = \sum \boldsymbol{A}$ for suitable sets $\boldsymbol{A} \in$

$(\mathcal{C}M)^k$. Since f is a $\mathcal{C}$-morphism, hence a dioid-homomorphism, we obtain

$$\begin{aligned} f(\mu \boldsymbol{x}\boldsymbol{p}^M(\sum \boldsymbol{A})) &= f(\sum \mu \boldsymbol{x}\boldsymbol{p}^{\mathcal{C}M}(\boldsymbol{A})) = \sum \widetilde{f}(\mu \boldsymbol{x}\boldsymbol{p}^{\mathcal{C}M}(\boldsymbol{A})) \\ &= \sum (\mu \boldsymbol{x}\boldsymbol{p}^{\mathcal{C}M'}(\widetilde{f}(\boldsymbol{A}))) = \mu \boldsymbol{x}\boldsymbol{p}^{\mathcal{C}M'}(\sum \widetilde{f}(\boldsymbol{A})) \\ &= \mu \boldsymbol{x}\boldsymbol{p}^{\mathcal{C}M'}(f(\sum \boldsymbol{A})). \end{aligned}$$

$\Leftarrow$: Let f be a semiring homomorphism that preserves least solutions of polynomial inequations. We show $f(\sum U) = \sum \widetilde{f}(U)$, by induction on $U \in \mathcal{C}M$. It is clear for $U \in \mathcal{F}M$. Otherwise, U is a component of the least solution $\boldsymbol{U}$ of some polynomial system $\boldsymbol{x} \geq \boldsymbol{p}^{\mathcal{C}M}(\boldsymbol{x}, \boldsymbol{A})$, where $f(\sum \boldsymbol{A}) = \sum \widetilde{f}(\boldsymbol{A})$ for the parameters $\boldsymbol{A} \in \mathcal{C}M$ by induction. From the proof of Theorem 3 we know $\sum \boldsymbol{U} = \sum \mu \boldsymbol{x}\boldsymbol{p}^{\mathcal{C}M}(\boldsymbol{A}) = \mu \boldsymbol{x}\boldsymbol{p}^M(\sum \boldsymbol{A})$, so by the assumption on f

$$\begin{aligned} f(\sum \boldsymbol{U}) &= f(\mu \boldsymbol{x}\boldsymbol{p}^M(\sum \boldsymbol{A})) = \mu \boldsymbol{x}\boldsymbol{p}^{M'}(f(\sum \boldsymbol{A})) \\ &= \mu \boldsymbol{x}\boldsymbol{p}^{M'}(\sum \widetilde{f}(\boldsymbol{A})) = \sum \mu \boldsymbol{x}\boldsymbol{p}^{\mathcal{C}M'}(\widetilde{f}(\boldsymbol{A})) \\ &= \sum \widetilde{f}(\mu \boldsymbol{x}\boldsymbol{p}^{\mathcal{C}M}(\boldsymbol{A})) = \sum \widetilde{f}(\boldsymbol{U}). \end{aligned}$$

6 Conclusion

We have shown that the categories of C-dioids of Hopkins [6] and of μ-continuous Chomsky algebras of Grathwohl et al. [3] coincide. To do so, we have replaced the somewhat technical grammar-based definition of context-free subsets $\mathcal{C}M$ of a monoid M from Hopkins [6] by a more natural, but equivalent definition as the closure of the collection of finite subsets of M under least solutions of polynomial inequations with parameters from $\mathcal{C}M$, which avoids a detour through free monoids. Our proofs exhibit a direct correspondence between the stages of the construction of least solutions $\boldsymbol{U} \in (\mathcal{C}M)^n$ of polynomial inequations and their least upper bounds $\sum \boldsymbol{U} \in M^n$, as was to be expected.

If one is not interested in this correspondence, one can obtain Theorem 3, as pointed out by a reviewer, from Lemma 3.1 of Grathwohl et al. [3], which "asserts that the supremum of a context-free language over a μ-continuous Chomsky algebra K exists, interpreting strings over K as products in K. Moreover, multiplication is continuous with respect to context-free languages." And given that for a $\mathcal{C}$-dioid M the sets in $\mathcal{C}M$ are the context-free sets in the sense just indicated and have least upper bounds denoted by μ-terms, the $\mathcal{C}$-distributivity amounts to the μ-continuity property by Corollary 1, which roughly gives Theorem 2.

By the equivalence of $\mathcal{C}$-dioids and μ-continuous Chomsky algebras, we can transfer closure under coproducts, coequalizers and tensor products from the former to the latter, and transfer axiomatizablility and completeness results from the latter to the former.

Let us close with an open question. A polynomial with parameters from a monoid M is *linear*, if each of its monomials has at most one variable factor. Let $\mathcal{L}M$, the *metalinear subsets* of M, be the closure of $\mathcal{F}M$ under (binary) union and finite products of components of least solutions of systems $\boldsymbol{x} \geq \boldsymbol{p}(\boldsymbol{x})$ of linear polynomials with parameters from M (cf. Harrison [5], p.64, for the metalinear languages). Similar to the proof of Theorem 1, we can show that $\mathcal{L}$ is a monadic operator, thereby obtaining a category $\mathbb{D}\mathcal{L}$ of $\mathcal{L}$-dioids and $\mathcal{L}$-morphisms. On the other hand, call a dioid M *linear-algebraically closed* if each system of linear polynomial inequations has least solutions. Let the *linear-μ-terms* be those that arise as solution terms of linear polynomial inequations through Bekić's reduction. It seems that our proofs of Theorem 2 and Theorem 3 can be specialized to show that the $\mathcal{L}$-dioids and the linear-algebraically closed, linear-μ-continuous dioids are the same. Moreover, can this be continued by specializing the completeness theorem of Grathwohl et al.[3] to provide a complete axiomatization of the equational theory of metalinear languages, consisting of the dioid axioms and a linear-μ-continuity axiom?

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