# $C$-Dioids = $\mu$-Continuous Chomsky Algebras


Hans Leiß
leiss@cis.uni-muenchen.de

Universität München
Centrum für Informations- und Sprachverarbeitung


Oberseminar Theoretische Informatik, LMU, 22.6.2018

# Abstract


Title: C-dioids and $\mu$-continuous Chomsky-algebras

In their complete axiomatization of the equational theory of context-free languages, Grathwohl, Henglein and Kozen (FICS 2013) introduced $\mu$-continuous Chomsky algebras. These are algebraically complete idempotent semirings where multiplication and the least-fixed-point operator $\mu$ are related by a continuity condition.

In his algebraic generalization of the Chomsky hierarchy, Hopkins (RelMiCS 2008) introduced C-dioids, which are idempotent semirings (or: dioids) where context-free subsets have least upper bounds and multiplication is sup-continuous.

We show that these two classes of structures coincide.

# Content

1. Chomsky-algebras: idempotent semirings $(M, +, \cdot, 0, 1)$ in which CFGs $\bar{x} \geq \bar{p}(\bar{x})$ have least solutions $\mu\bar{x}\bar{p}^M$.
   - $\mu$-continuity: $a \cdot \mu xt^M \cdot b = \sum\{a \cdot mxt^M \cdot b \mid m \in \mathbb{N}\}$

2. C-dioids: idempotent semirings $(M, +, \cdot, 0, 1)$ with
   - sups $\sum U \in M$ of context-free subsets $U \subseteq M$
   - sup-continuity: $(\sum U)(\sum V) = \sum(UV)$ for cf-sets $U, V$.

We show:

$$\mu\text{-continuous Chomsky algebra} = \text{C-dioid}.$$

# 0. Definitions: $\mathcal{A}$-dioids, Kleene and Chomsky algebras

A semiring $R = (R, +, 0, \cdot, 1)$ is a set $R$ with two operations $+, \cdot : R \times R \to R$, such that $(R, +, 0)$ and $(R, \cdot, 1)$ are monoids, $+$ is commutative, and the zero and distributivity laws holds:

$$\forall a, b, c, d : \quad a0b = 0, \quad a(b + c)d = abd + acd$$

A dioid or idempotent semiring $D = (D, +, 0, \cdot, 1)$ is a semiring in which $+$ is idempotent. It has a natural partial order $\leq$, defined by

$$a \leq b :\iff a + b = b.$$

A partially ordered monoid $(M, \cdot, 1, \leq)$ is a monoid $(M, \cdot, 1)$ with a partial order $\leq$ and where $\cdot$ is monotone in each argument.

If $M = (M, \cdot^M, 1^M)$ is a monoid, its power set $(\mathcal{P}(M), \cdot, 1, \subseteq)$ is a partially ordered monoid –and $(\mathcal{P}(M), \cup, \emptyset, \cdot, 1)$ a dioid–, where

$$A \cdot B := \{a \cdot^M b \mid a \in A, b \in B\}, \quad 1 := \{1^M\}.$$

A functor $\mathcal{A} : \text{Monoid} \to \text{Monoid}$ is monadic (Hopkins[3]), if for each monoid $M$

- $A_0$ $\mathcal{A}M$ is a set of subsets of $M$: $\mathcal{A}M \subseteq \mathcal{P}M$,
- $A_1$ $\mathcal{A}M$ contains each finite subset of $M$: $\mathcal{F}M \subseteq \mathcal{A}M$,
- $A_2$ $\mathcal{A}M$ is closed under product (hence a monoid),
- $A_3$ $\mathcal{A}M$ is closed under union of sets from $\mathcal{A}\mathcal{A}M$, and
- $A_4$ $\mathcal{A}M$ preserves monoid-homomorphisms: if $f : M \to N$ is a homomorphism, so is $\widetilde{f} : \mathcal{A}M \to \mathcal{A}N$, where for $U \subseteq M$

$$\widetilde{f}(U) := \{f(u) \mid u \in U\}.$$

Theorem (Hopkins[3]): The monadic functors form a lattice.

Example (algebraic Chomsky' hierarchy)

The functors $\mathcal{F} \leq \mathcal{R} \leq \mathcal{L} \leq \mathcal{C} \leq \mathcal{T} \leq \mathcal{P}$ are monadic ($A_3$!):

1. $\mathcal{P}M$ = all subsets of $M$,
2. $\mathcal{F}M$ = all finite subsets of $M$,
3. $\mathcal{R}M$ = the closure of $\mathcal{F}M$ under $+$ (union), $\cdot$ (elementwise product) and $^*$ (iteration), i.e. $A^* = \bigcup\{A^n \mid n \in \mathbb{N}\}$.
4. $\mathcal{L}M$ = the closure of $\mathcal{F}M$ under $+$ and products of least solutions in $\mathcal{P}M$ of $x \geq p(x)$ with linear polynomials $p(x)$ over $\mathcal{L}M$, i.e. $p(x) = a_1 x b_1 + \ldots a_k x b_k + c$ with $a_i, b_i, c \in \mathcal{L}M$.
5. $\mathcal{C}M$ = the closure of $\mathcal{F}M$ under least solutions in $\mathcal{P}M$ of systems $x_1 \geq p_1(\bar{x}), \ldots x_n \geq p_n(\bar{x})$ with polynomials $p_i(\bar{x})$ over $\mathcal{C}M$.
6. $\mathcal{T}M$ = all Turing/Thue-subsets $\mathcal{T}M$ of $M$.

Rem. $\mathcal{S}M$ = all context-sensitive subsets of $M$ is *not* monadic.($A_4$)

Let $M$ be a partially ordered monoid. For $a \in M$ and $U \subseteq M$ let $U < a$ mean that $a$ is an upper bound of $U$: for all $u \in U$, $u \leq a$.

- $D_0$ $M$ is $\mathcal{A}$-complete, if each $U \in \mathcal{A}M$ has a least upper bound $\sum U \in M$.
- $D_1$ $M$ is $\mathcal{A}$-continuous, if for all $U \in \mathcal{A}M$ and $x, a, b \in M$ with $x > aUb$ there is some $u > U$ with $x \geq aub$.

Prop. (Hopkins 2008) If the partially ordered monoid $M$ is $\mathcal{A}$-complete, the conditions $D_1, D_1', D_2'$ are pairwise equivalent:

- $D_1'$ for all $a, b \in M$ and $U \in \mathcal{A}M$, $\sum aUb = a(\sum U)b$.
- $D_2'$ for all $U, V \in \mathcal{A}M$, $\sum(UV) = \sum U \cdot \sum V$.

These are called *weak* resp. *strong* $\mathcal{A}$-distributivity.

Clearly, $D_2' \Rightarrow D_1'$. We later need a local version of $D_1' \Rightarrow D_2'$:

**Prop.** Let $M$ be a partially ordered monoid and $U, V \in \mathcal{A}M$ such that $u := \sum U$ and $v := \sum V$ exist. Then (i) implies (ii) for

(i) for all $a, b \in M$, $\sum aUb = a(\sum U)b$ and $\sum aVb = a(\sum V)b$.
(ii) $\sum(UV) = \sum U \cdot \sum V$.

**Proof.**

Clearly, $UV < uv$. To prove that $uv$ is $\sum(UV)$, take any $c \in M$ with $UV < c$ and show $uv \leq c$.

For each $a \in U$, by (i), $\sum aV1$ exists, and as $aV1 \subseteq UV < c$,

$$av = a(\sum V)1 = \sum aV1 \leq c.$$

Hence $Uv = 1Uv < c$.

By (i), $\sum 1Uv$ exists, and $uv = 1(\sum U)v = \sum 1Uv \leq c$. □

An $\mathcal{A}$-dioid is a partially ordered monoid $M$ which is

- $D_0$ $\mathcal{A}$-complete: every $U \in \mathcal{A}M$ has a supremum $\sum U \in M$, and
- $D_2'$ $\mathcal{A}$-distributive: for all $U, V \in \mathcal{A}M$, $\sum(UV) = (\sum U)(\sum V)$.

Every $\mathcal{A}$-dioid $(M, \cdot, 1, \leq)$ is a dioid, using $a + b := \sum\{a, b\}$ and $0 := \sum \emptyset$. The zero and distributivity laws follow from $D_1' \equiv D_2'$.

### Lemma

*If $M$ is an $\mathcal{A}$-dioid and $p(x_1, \ldots, x_n)$ a polynomial in $x_1, \ldots, x_n$ with parameters from $M$, then $p^{\mathcal{A}M}(U_1, \ldots, U_n) \in \mathcal{A}M$ for all $U_1, \ldots, U_n \in \mathcal{A}M$ –with $m^{\mathcal{A}M} := \{m\}$ for $m \in M$–, and*

$$\sum p^{\mathcal{A}M}(U_1, \ldots, U_n) = p^M(\sum U_1, \ldots, \sum U_n).$$

Proof.

This follows from $\sum\{m\} = m$, $\mathcal{A}$-distributivity and

$$\sum(U+V) = \sum U + \sum V \quad \text{for all } U, V \in \mathcal{A}M.$$

Since $\{U, V\} \in \mathcal{F}\mathcal{A}M \subseteq \mathcal{A}\mathcal{A}M$, $U + V = \bigcup\{U, V\} \in \mathcal{A}M$, and so there is a least upper bound $\sum(U+V) \in M$. Hence

$$\sum U + \sum V \leq \sum(U+V) + \sum(U+V) = \sum(U+V).$$

As $U + V < \sum U + \sum V$, so $\sum(U+V) \leq \sum U + \sum V$. $\square$ $\square$

The monadic operator $\mathcal{A}$ provides us with a notion of continuous maps between partially ordered monoids, as follows.

$D_3$ A map $f : M \to M'$ is $\mathcal{A}$-continuous, if for all $U \in \mathcal{A}M$ and $y > \widetilde{f}(U)$ there is some $x > U$ with $y \geq f(x)$.

An $\mathcal{A}$-morphism is a $\leq$-preserving, $\mathcal{A}$-continuous homomorphism.

Let $D\mathcal{A}$ be the category of $\mathcal{A}$-dioids with $\mathcal{A}$-morphisms.

Every $\mathcal{A}$-morphism between $\mathcal{A}$-dioids is a dioid-homomorphism.

An $\leq$-preserving homomorphism $f : M \to M'$ between $\mathcal{A}$-dioids is $\mathcal{A}$-continuous iff

$$f(\sum U) = \sum \widetilde{f}(U) \qquad \text{forall } U \in \mathcal{A}M.$$

## Theorem

- *(Hopkins 2008)* $\mathcal{A}M$ *is the free* $\mathcal{A}$*-dioid with generators* $M$.
- *(Hopkins 2008)* $D\mathcal{A}$ *has a tensor product* $D \otimes_{\mathcal{A}} D'$, *satisfying*

$$\mathcal{A}M \otimes_{\mathcal{A}} \mathcal{A}M' \simeq \mathcal{A}(M \times M').$$

  - $\mathcal{R}(M \times M') =$ *rational transductions between* $M$ *and* $M'$.
  - $\mathcal{C}(M \times M') =$ *simple syntax-directed translations btw* $M, M'$.
- *(HL 2018)* $D\mathcal{A}$ *has co-products* $D \oplus_{\mathcal{A}} D'$ *and co-equalizers (quotients by* $\mathcal{A}$*-congruences), hence co-limits.*

## Theorem (Hopkins 2008)

$D\mathcal{R}$ *equals Kozen's category of* $^*$*-continuous Kleene algebras.*

## Kozen 1981/1990: $^*$-continuous Kleene-algebras

A Kleene algebra $(K, +, 0, \cdot, 1, ^*)$ is an idempotent semiring (dioid) $(K, +, 0, \cdot, 1)$ with a unary operation $^* : K \to K$ such that

- (KA 1) $\forall a, b \in K :$ $a^* b$ is the least solution of $x \geq ax + b$.
- (KA 2) $\forall a, b \in K :$ $ba^*$ is the least solution of $x \geq xa + b$.

The Kleene algebra $K$ is $^*$-continuous, if for all $a, b, c \in K$,

$$ac^* b = \sum \{ac^n b \mid n \in \mathbb{N}\}.$$

In particular:

- $K$ is $^*$-complete: every set $U_c = \{c^n \mid n \in \mathbb{N}\}$ has a supremum, $c^* = \sum U_c$.
- $\cdot$ is $^*$-distributive: for all $a, b, c$, $a(\sum U_c)b = \sum(aU_c b)$.

# $\mathcal{C}$-dioids

We are interested in the category $D\mathcal{C}$ of $\mathcal{C}$-dioids as a generalization of the theory of context-free languages over free monoids.

Why consider $\mathcal{C}M \subseteq \mathcal{P}M$ for non-free monoids $M$?

- We want to handle transductions $T \subseteq X^* \times Y^*$ in the same formalism as we handle languages, but $X^* \times Y^*$ is not free: for example, $(x, \epsilon)(\epsilon, y) = (x, y) = (\epsilon, y)(x, \epsilon)$.
- Natural languages apply "sound laws" to concatenate stem+affix in a non-free way: *bet+ing* = *betting*
- Natural languages apply inflections to concatenate words and phrases in a non-free way: *few* + *man* = *few men*, *this woman* + *(to) read a book* = *this woman reads a book*.

## Claim

*$D\mathcal{C}$ equals the category of $\mu$-continuous Chomsky-algebras.*

## Partially ordered $\mu$-semirings

Let $X$ be an infinite set of variables and consider $\mu$-terms over $X$:

$$s, t := x \mid 0 \mid 1 \mid (s \cdot t) \mid (s + t) \mid \mu x\, t$$

A partially ordered $\mu$-semiring $(M, +, \cdot, 0, 1, \leq)$ is a semiring $(M, +, \cdot, 0, 1)$ with a partial order $\leq$ on $M$, where every term $t$ defines a function $t^M : (X \to M) \to M$, so that

for all terms $s, t$, $x \in X$ and valuations $g, h : X \to M$

1. $$\begin{array}{lcl lcl} 0^M(g) & = & 0, & (s+t)^M(g) & = & s^M(g) + t^M(g), \\ 1^M(g) & = & 1, & (s \cdot t)^M(g) & = & s^M(g) \cdot t^M(g), \\ x^M(g) & = & g(x), & \text{if } s^M \leq t^M, & \text{then} & \mu x s^M \leq \mu x t^M, \end{array}$$
2. $t^M(g) \leq t^M(h)$, if $g \leq h$ pointwise,
3. $t^M(g) = t^M(h)$, if $g = h$ on $free(t)$, (coincidence prop.)
4. $t[x/s]^M(g) = t^M(g[x/s^M(g)])$. (substitution prop.)

For $t(x_1, \ldots, x_n)$ we write $t^M[x_1/a_1, \ldots, x_n/a_n]$ or $t^M(a_1, \ldots, a_n)$.

A Park $\mu$-semiring is a partially ordered $\mu$-semiring $M$ where for all terms $t$ and variables $x, y$, the following hold in $M$:

(Park axiom) $t[x/\mu xt] \leq \mu xt$,
(Park rule) $t[x/y] \leq y \rightarrow \mu xt \leq y$.

In a Park $\mu$-semiring $M$, $\mu xt^M(g)$ is the least solution of $t \leq x$ in $M, g$, i.e. the least $a \in M$ such that $t^M(g[x/a]) \leq a$.

From the Park axiom and rule, it follows easily that

$$t[x/\mu xt] = \mu xt, \quad \text{and} \quad \mu y.t[x/y] = \mu xt \text{ for } y \notin free(t),$$

hold in $M$.

# Kozen e.a. 2013: $\mu$-continuous Chomsky-algebras

An idempotent semiring $(M, +, 0, \cdot, 1)$ is algebraically closed or a Chomsky-algebra, if every system

$$x_1 \geq p_1(\bar{x}, \bar{y}), \ldots, x_n \geq p_n(\bar{x}, \bar{y}), \quad \bar{x} = x_1, \ldots, x_n, n \in \mathbb{N},$$

with polynomials $p_i(\bar{x}, \bar{y})$ has least solutions $\bar{a} \in K^n$, for all parameters $\bar{b} \in K^m$ for $\bar{y} = y_1, \ldots, y_m$.

## Example

The set $\mathcal{C}X^*$ of context-free languages over $X$ is the smallest set $\mathcal{L} \subseteq \mathcal{P}X^*$ such that

(i) each finite subset of $X \cup \{\epsilon\}$ is in $\mathcal{L}$, and

(ii) if $\bar{x} \geq \bar{p}(\bar{x}, \bar{y})$ is a polynomial system, and $\bar{B} \in \mathcal{L}^m$, then the the least $\bar{A} \in (\mathcal{P}X^*)^n$ with $\bar{A} \supseteq \bar{p}^{\mathcal{P}X^*}(\bar{A}, \bar{B})$ belongs to $\mathcal{L}^n$.

Then $(\mathcal{C}X^*, +, \cdot, 0, 1)$ is a Chomsky algebra. [Least solutions of $\bar{x} \geq \bar{p}(\bar{x}, \bar{y})$ exist in $\mathcal{P}X^*$, as this is a CPO and $+, \cdot$ are continuous.]

## Lemma (Grathwohl,Henglein,Kozen (FICS 2013))

*Every Chomsky-algebra $M$ is an idempotent, partially ordered $\mu$-semiring, if we define for terms $t$, $x \in X$ and $g : X \to M$*

$$\mu x t^M(g) := \text{the least } a \in M \text{ such that } t^M(g[x/a]) \leq a. \qquad (1)$$

*Moreover, every system $\bar{t}(\bar{x}, \bar{y}) \leq \bar{x}$ with $\mu$-terms $\bar{t}(\bar{x}, \bar{y})$ has least solutions in $M$, i.e. for all parameters $\bar{b}$ from $M$ there is a least tuple $\bar{a}$ in $M$ such that $\bar{t}^M(\bar{a}, \bar{b}) \leq \bar{a}$.*

Proof: by reduction to least solutions of polynomial systems.

## Corollary

*Every Chomsky algebra is a Park $\mu$-semiring (using these $\mu x t^M$).*

A Chomsky algebra $M$ is $\mu$-continuous, if for all $a, b \in M$, all terms $t$, $x \in X$ and $g : X \to M$ it satisfies

$$a \cdot \mu x t^M(g) \cdot b = \sum \{a \cdot m x t^M(g) \cdot b \mid m \in \mathbb{N}\}, \qquad (2)$$

where $mxt$ is defined by $0xt := 0$, $(m+1)xt := t[x/mxt]$.

The $\mu$-continuity condition generalizes Kozen's $^*$-continuity

$$a \cdot c^* \cdot b = \sum \{a \cdot c^m \cdot b \mid m \in \mathbb{N}\}.$$

### Theorem (Grathwohl,Henglein,Kozen, 2013)

*For terms $s, t$ are equivalent:*

- $s^{\mathcal{P}X^*}(g) = t^{\mathcal{P}X^*}(g)$ *for the standard valuation* $g(x) = \{x\}$,
- $s^M(g) = t^M(g)$ *for all $\mu$-continuous CAs $M$ and* $g : X \to M$.

# I. Every $\mu$-continuous Chomsky algebra is a $\mathcal{C}$-dioid

We first define term vectors $\mu\bar{x}\bar{t}$ that embody H. Bekić's (1984) reduction of the $n$-ary least fixed-point operator to the unary one *in $\omega$-complete partial orders with sup-continuous operations*.

For vectors $\bar{t} = t_1, \ldots, t_n$ of terms and $\bar{x} = x_1, \ldots, x_n$ of pairwise different variables, define the term vector $\mu\bar{x}\bar{t}$ as follows. If $n = 1$, then $\mu\bar{x}\bar{t} := \mu x_1 t_1$. If $n > 1$, $\bar{x} = (\bar{y}, \bar{z})$ and $\bar{t} = (\bar{r}, \bar{s})$ with term vectors $\bar{r}, \bar{s}$ of lengths $|\bar{y}|, |\bar{z}| < n$, then $\mu\bar{x}\bar{t}$ is

$$\mu(\bar{y}, \bar{z})(\bar{r}, \bar{s}) := (\mu\bar{y}.\bar{r}[\bar{z}/\mu\bar{z}\bar{s}], \mu\bar{z}.\bar{s}[\bar{y}/\mu\bar{y}\bar{r}]). \qquad (3)$$

## Lemma (HL[4])

*For any Chomsky algebra $M$ and valuation $g : X \to M$ is $\mu\bar{x}\bar{t}^M(g)$ the least tuple $\bar{a}$ in $M$ such that $\bar{t}^M(g[\bar{x}/\bar{a}]) \leq \bar{a}$.*

The value $\mu\bar{x}\bar{t}^M(g)$ does not depend on the splitting $\bar{x}$ into $\bar{y}, \bar{z}$.

The unary version of $\mu$-continuity implies the $n$-ary version:

## Lemma (Cor. 23 in [4])

*Let $M$ be a $\mu$-continuous Chomsky algebra and $g : X \to M$. Then*

$$\bar{a} \cdot \mu\bar{x}\bar{t}^M(g) \cdot \bar{b} = \sum\{\bar{a} \cdot m\bar{x}\bar{t}^M(g) \cdot \bar{b} \mid m \in \mathbb{N}\},$$

*for any term vector $\bar{t}$ and $\bar{a}, \bar{b} \in M^{|t|}$, and $(m+1)\bar{x}\bar{t} := \bar{t}[\bar{x}/m\bar{x}\bar{t}]$.*

## Theorem

*Let $M$ be a $\mu$-continuous Chomsky-algebra. Then $M$ is a $\mathcal{C}$-dioid:*

a) *Every $U \in \mathcal{C}M$ has a supremum $\sum U \in M$ ($\mathcal{C}$-completeness).*
b) *For all $U, V \in \mathcal{C}M$, $\sum(UV) = (\sum U)(\sum V)$ ($\mathcal{C}$-distributivity)*

*Proof.* As $M$ is a dioid, a) and b) are true for all $U, V \in \mathcal{F}M$.

Let $\bar{U} \in (\mathcal{C}M)^n$ be the least solution of $\bar{x} \geq \bar{p}^{\mathcal{C}M}(\bar{x}, \bar{A})$. By induction, we may assume a) and b) for all $U, V \in \bar{A}$. To show them for all $U, V \in \bar{U}, \bar{A}$, by a previous Prop. we only need:

a') Every $U \in \bar{U}$ has a supremum $\sum U \in M$.
b') For all $U \in \bar{U}$ and all $a, b \in M$, $\sum(aUb) = a(\sum U)b$.

Notice that b) for all $U, V \in \bar{A} \cup \mathcal{F}M$ gives us b') for all $U \in \bar{A}$.

Idea: There is a least solution $\bar{u} \in M^n$ of $\bar{x} \geq \bar{p}^M(\bar{x}, \bar{a})$, which ought to give sup's for $\bar{U} = \mu\bar{x}\bar{p}^{\mathcal{C}M}(\bar{A})$, hence $\sum U$ should exist by

$$\sum \bar{U} = \sum \mu\bar{x}p^{\mathcal{C}M}(\bar{A}) = \mu\bar{x}\bar{p}^M(\sum A) = \bar{u},$$

which in turn must come from $\sum \bar{U}_m = \bar{u}_m$ of its approximations

$$\bar{U}_m = m\bar{x}\bar{p}^{\mathcal{C}M}(\bar{x}, \bar{A}) \quad \text{and} \quad \bar{u}_m = m\bar{x}\bar{p}^M(\bar{x}, \sum \bar{A}).$$

To show $\sum \bar{U}_m = u_m$ inductively, we need $\mathcal{C}$-distributivity of $\bar{U}_m, \bar{A}$:

Consider $x \geq p(x, y, z) := yx + z$. Suppose $A, B \in \mathcal{C}M$ have least upper bounds $\sum A = a$, $\sum B = b \in M$. Since $M$ is $\mu$-continuous,

$$\mu xp^M(a, b) = a^* b = \sum\{a^m b \mid m \in \mathbb{N}\}.$$

To show that $(m+1)xp^M(a, b) = a \cdot mxp^M(a, b) + b$ is the least upper bound of $(m+1)xp^{\mathcal{C}M}(A, B) = A \cdot mxp^{\mathcal{C}M}(A, B) \cup B$, we need to know a case of (strong) $\mathcal{C}$-distributivity:

$$\begin{aligned} a \cdot mxp^M(a, b) + b &= (\sum A)(\sum mxp^{\mathcal{C}M}(A, B)) + \sum B \\ &= \sum(A \cdot mxp^{\mathcal{C}M}(A, B) \cup B). \end{aligned}$$

By induction, we prove for $\bar{U}_m := m\bar{x}\bar{p}^{\mathcal{C}M}(\bar{A})$, $\bar{u}_m := m\bar{x}\bar{p}^M(\sum \bar{A})$

(i) $\sum(\bar{U}_m, \bar{A})$ exists (componentwise),

(ii) for all monomials $q(\bar{x}, \bar{y})$, $q^M(\sum \bar{U}_m, \sum \bar{A}) = \sum q^{\mathcal{C}M}(\bar{U}_m, \bar{A})$,

(iii) $\bar{u}_m = \sum \bar{U}_m$.

For $m = 0$, (iii) is clear: $\bar{0} = \sum \bar{\emptyset}$. Therefore, (i) and (ii) follow from the hypothesis a') $\sum \bar{A}$ exist and b') distributivity for $\bar{A}$;

(ii) extends to polynomials by $\sum(A \cup B) = \sum A + \sum B$.

For $m+1$, by induction $\sum \bar{U}_m$ exists by $(i)$, and then

$$\begin{aligned}
\bar{u}_{m+1} &= \bar{p}^M(\bar{u}_m, \textstyle\sum \bar{A}) && \text{(def.)} \\
&= \bar{p}^M(\textstyle\sum \bar{U}_m, \sum \bar{A}) && \text{(iii)} \\
&= \textstyle\sum \bar{p}^{\mathcal{C}M}(\bar{U}_m, \bar{A}) && \text{(ii)} \\
&= \textstyle\sum \bar{U}_{m+1} && \text{(def.)}
\end{aligned}$$

Hence, (i) $\sum \bar{U}_{m+1}$ exists, and (iii) $\bar{u}_{m+1} = \sum \bar{U}_{m+1}$.

For (ii), let $q(\bar{x}, \bar{y})$ be a monomial in $\bar{x}, \bar{y}$, and $r(\bar{x}, \bar{y})$ the polynomial obtained by distribution from $q(\bar{x}, \bar{y})[\bar{x}/\bar{p}(\bar{x}, \bar{y})]$. Then

$$\begin{aligned}
q^M(\sum \bar{U}_{m+1}, \sum \bar{A}) &= r^M(\sum \bar{U}_m, \sum \bar{A}) \\
&= \sum r^{\mathcal{C}M}(\bar{U}_m, \bar{A}) && \text{((ii) for } r) \\
&= \sum q^{\mathcal{C}M}(\bar{U}_{m+1}, \bar{A}).
\end{aligned}$$

Now $\bar{u} := \mu\bar{x}\bar{p}^M(\sum \bar{A})$ is the least upper bound of $\bar{U} = \mu\bar{x}\bar{p}^{\mathcal{C}M}(\bar{A})$:

$$
\begin{aligned}
\bar{u} &= \mu\bar{x}\bar{p}^M(\textstyle\sum \bar{A}) \\
&= \textstyle\sum\{m\bar{x}\bar{p}^M(\sum \bar{A}) \mid m \in \mathbb{N}\} \quad (M \text{ a } \mu\text{-cont.CA}) \\
&= \textstyle\sum\{\bar{u}_m \mid m \in \mathbb{N}\} \\
&= \textstyle\sum\{\sum \bar{U}_m \mid m \in \mathbb{N}\} \qquad \text{(iii)} \\
&= \textstyle\sum \bigcup\{\bar{U}_m \mid m \in \mathbb{N}\} \\
&= \textstyle\sum \bar{U} = \sum \mu\bar{x}\bar{p}^{\mathcal{C}M}(\bar{A}).
\end{aligned}
$$

In particular, we have shown a') any $U \in \bar{U}$ has a $\sum U \in M$.

To show b') $a(\sum U)b = \sum(aUb)$, extend $a, b$ to some $\bar{a}, \bar{b} \in M^n$.

Having $\bar{a}(\sum \bar{U}_m)\bar{b} = \sum \bar{a}\bar{U}_m\bar{b}$ inductively by (ii), we obtain

$$
\begin{aligned}
\bar{a}(\sum \bar{U})\bar{b} &= \bar{a} \cdot \bar{u} \cdot \bar{b} \\
&= \sum\{\bar{a} \cdot \bar{u}_m \cdot \bar{b} \mid m \in \mathbb{N}\} && (M\ \mu\text{-cont.CA}) \\
&= \sum\{\bar{a}(\sum \bar{U}_m)\bar{b} \mid m \in \mathbb{N}\} && (\bar{u}_m = \sum \bar{U}_m) \\
&= \sum\{\sum(\bar{a}U_m\bar{b}) \mid m \in \mathbb{N}\} && (\text{by (ii)}) \\
&= \sum \bigcup\{\bar{a}\bar{U}_m\bar{b} \mid m \in \mathbb{N}\} && (\textstyle\sum \text{ property}) \\
&= \sum(\bar{a} \cdot \bigcup\{\bar{U}_m \mid m \in \mathbb{N}\} \cdot \bar{b}) && (\cdot^{\mathcal{C}M} \text{ is } \textstyle\bigcup\text{-cont.}) \\
&= \sum(\bar{a}\bar{U}\bar{b}).
\end{aligned}
$$

Hence, for $U \in \bar{U}$ we have b') $a(\sum U)b = \sum aUb$ for all $a, b$. □

# II. Every $\mathcal{C}$-dioid is a $\mu$-continuous Chomsky algebra

## Theorem

*Let $M$ be a $\mathcal{C}$-dioid. Then $M$ be a $\mu$-continuous Chomsky-algebra.*

*Proof.* (i) $M$ is algebraically closed: Let $\bar{x} \geq \bar{p}(\bar{x}, \bar{y})$ be a polynomial system with $n = |\bar{x}|, k = |\bar{y}|$, and $\bar{a} \in M^k$. Let $\bar{A}$ consist of the $A_j := \{a_j\} \in \mathcal{C}M$, so $\bar{a} = \sum \bar{A}$, and let

$$\bar{U} = \mu\bar{x}\bar{p}^{\mathcal{C}M}(\bar{A}) \in (\mathcal{C}M)^n$$

be the least solutuion of $\bar{x} \geq \bar{p}^{\mathcal{P}M}(\bar{x}, \bar{A})$ in $\mathcal{P}M$.

Since $M$ is a $\mathcal{C}$-dioid, suprema $u_i := \sum U_i \in M$ exist. We show that $\bar{u} := \sum \bar{U}$ is the least solution of $\bar{x} \geq \bar{p}^M(\bar{x}, \bar{b})$ in $M$, i.e.

$$\mu\bar{x}\bar{p}^M(\bar{a}) = \bar{u} = \sum \bar{U} = \sum \mu\bar{x}\bar{p}^{\mathcal{C}M}(\bar{A}). \qquad (4)$$

Since $M$ is $\mathcal{C}$-distributive, $\bar{u} = \sum \bar{U}$ is a solution of $\bar{x} \geq \bar{p}^M(\bar{x}, \bar{a})$:[1]

$$\bar{p}^M(\sum \bar{U}, \sum \bar{A}) = \sum \bar{p}^{\mathcal{C}M}(\bar{U}, \bar{A}) \ \leq \ \sum \bar{U}.$$

To show that $\bar{u}$ is the least solution of $\bar{x} \geq \bar{p}^M(\bar{x}, \bar{a})$, let $\bar{c} \in M^n$ be any solution. It is sufficient to show $\bar{c} > \bar{U}$. We know

$$\bar{U} = \bigcup \{\bar{p}^{\mathcal{P}M}(\bar{U}_m, \bar{A}) \mid m \in \mathbb{N}\}$$

where $\bar{U}_0 := \bar{\emptyset}$, $\bar{U}_{m+1} := \bar{p}^{\mathcal{P}M}(\bar{U}_m, \bar{A})$.

For $m = 0$, obviously $\bar{c} > \bar{U}_0$. Suppose $\bar{c} > \bar{U}_m$ for some $m$. By induction on $p_i$, $p_i^{\mathcal{C}M}(\bar{U}_m, \bar{A}) < p_i^M(\bar{c}, \bar{a})$ for each $i$, hence

$$\bar{U}_{m+1} < \bar{p}^M(\bar{c}, \bar{a}) \leq \bar{c}.$$

Therefore, $\bar{U} < \bar{c}$.

[1] by Lemma 1

(ii) $M$ is $\mu$-continuous: we need an auxiliary

*Claim.* For all $\mu$-terms $t(x_1, \ldots, x_n)$ and sets $A_1, \ldots, A_n \in \mathcal{C}M$,

$$t^M(\sum A_1, \ldots, \sum A_n) = \sum t^{\mathcal{C}M}(A_1, \ldots, A_n). \tag{5}$$

*Proof.* By induction on $t$. For $(r \cdot s)$, by the $\mathcal{C}$-distributivity of $M$:

$$\begin{aligned}
(r \cdot s)^M(\sum \bar{A}) &= r^M(\sum \bar{A}) \cdot^M s^M(\sum \bar{A}) \\
&= (\sum r^{\mathcal{C}M}(\bar{A})) \cdot^M (\sum s^{\mathcal{C}M}(\bar{A})) \\
&= \sum (r^{\mathcal{C}M}(\bar{A}) \cdot^{\mathcal{C}M} s^{\mathcal{C}M}(\bar{A})) \\
&= \sum (r \cdot s)^{\mathcal{C}M}(\bar{A}).
\end{aligned}$$

For $\mu xr$, by induction we have for $B = \mu xr^{\mathcal{C}M}(\bar{A}) \in \mathcal{C}M$

$$r^M(\sum \bar{A}, \sum B) = \sum r^{\mathcal{C}M}(\bar{A}, B) \leq \sum B,$$

so that

$$\mu xr^M(\sum \bar{A}) \leq \sum B = \sum \mu xr^{\mathcal{C}M}(\bar{A}).$$

The converse holds by induction on Kozen's well-ordering $\prec$ of $\mu$-terms. Assuming $\sum mxr^{\mathcal{C}M}(\bar{A}) = mxr^M(\sum \bar{A})$ for all $m$, we get

$$\begin{aligned}\sum \mu xr^{\mathcal{C}M}(\bar{A}) &= \sum \bigcup \{mxr^{\mathcal{C}M}(\bar{A}) \mid m \in \mathbb{N}\} \\ &= \sum \{\sum mxr^{\mathcal{C}M}(\bar{A}) \mid m \in \mathbb{N}\} \\ &= \sum \{mxr^M(\sum \bar{A}) \mid m \in \mathbb{N}\} \\ &\leq \mu xr^M(\sum \bar{A}). \qquad \triangleleft\end{aligned}$$

We can now show the $\mu$-continuity condition. Since $g : X \to M$ is $\sum \circ g'$ for some $g' : X \to \mathcal{C}M$, by $g(x) = \sum\{g(x)\}$, it reads:

*Claim.* For all $\mu$-terms $\mu x t(\bar{x})$, all $\bar{A} \in (\mathcal{C}M)^{|\bar{x}|}$ and $a, b \in M$:

$$a \cdot \mu x t^M(\sum \bar{A}) \cdot b = \sum\{a \cdot m x t^M(\sum \bar{A}) \cdot b \mid m \in \mathbb{N}\}.$$

*Proof.*

$$\begin{aligned}
& a \cdot \mu x t^M(\textstyle\sum \bar{A}) \cdot b \\
= \;& (\textstyle\sum\{a\})(\sum \mu x t^{\mathcal{C}M}(\bar{A}))(\sum\{b\}) && \text{(by (5))} \\
= \;& \textstyle\sum(\{a\} \cdot \mu x t^{\mathcal{C}M}(\bar{A}) \cdot \{b\}) && (M \text{ a } \mathcal{C}\text{-dioid}) \\
= \;& \textstyle\sum(\{a\} \cdot \bigcup\{m x t^{\mathcal{C}M}(\bar{A}) \mid m \in \mathbb{N}\} \cdot \{b\}) \\
= \;& \textstyle\sum(\bigcup\{\{a\} \cdot m x t^{\mathcal{C}M}(\bar{A}) \cdot \{b\} \mid m \in \mathbb{N}\}) \\
= \;& \textstyle\sum\{\sum(\{a\} \cdot m x t^{\mathcal{C}M}(\bar{A}) \cdot \{b\}) \mid m \in \mathbb{N}\} \\
= \;& \textstyle\sum\{(\sum\{a\}) \cdot (\sum m x t^{\mathcal{C}M}(\bar{A})) \cdot (\sum\{b\}) \mid m \in \mathbb{N}\} \\
= \;& \textstyle\sum\{a \cdot m x t^M(\sum \bar{A}) \cdot b \mid m \in \mathbb{N}\}. && \text{(by (5))} \quad \triangleleft \; \square
\end{aligned}$$

# Open Problems (M.Hopkins' program)

I. To cover $\mathcal{S}M =$ context-sensitive subsets of $M$, consider a subcategory of Monoid with *non-erasing* homomorphisms.

II. Construct an explicit adjunction $Q^{\mathcal{C}}_{\mathcal{R}} : D\mathcal{R} \rightleftarrows D\mathcal{C} : Q^{\mathcal{R}}_{\mathcal{C}}$ between the category $D\mathcal{R}$ of $^*$-continuous Kleene algebras and the category $D\mathcal{C}$ of $\mu$-continuous Chomsky algebras.

To get $Q^{\mathcal{C}}_{\mathcal{R}}(\mathcal{R}X^*)$, modify the Chomsky-Schützenberger theorem:

$$\mathcal{C}X^* = \{e(R \cap D) \mid R \in \mathcal{R}((X \dot{\cup} Y)^*)\}, \quad \text{where}$$

- $Y = \{b, d, p, q\}$ consist of two bracket pairs $b, d$ and $p, q$,
- $e : (X \cup Y)^* \to X^*$ is the bracket-erasing homomorphism,
- $D \subseteq (X \cup Y)^*$ the Dyck-language of well-bracketed strings.

This gives $\mathcal{C}X^* = Q(\mathcal{R}(X \cup Y)^*)$; improve it to $\mathcal{C}X^* = Q(\mathcal{R}X^*)$, then to $\mathcal{C}M = Q(\mathcal{R}M)$ for monoids $M$, then to $Q^{\mathcal{C}}_{\mathcal{R}} : D\mathcal{R} \to D\mathcal{C}$.

Partial result (Hopkins):

Take $C_2 := \mathcal{R}Y^*/\{bd = 1 = pq, bq = 0 = pd\} \in D\mathcal{R}$. Then

$$\mathcal{C}X^* \subseteq \mathcal{R}X^* \otimes_{\mathcal{R}} C_2.$$

Example: In $C_2$, $bp^nq^md = 1$ if $n = m$, else 0. Hence

$$\begin{aligned} \mathcal{C}X^* &\ni \{x_1^n x_2^n \mid n \in \mathbb{N}\} \\ &\simeq \sum_n x_1^n x_2^n = \sum_{n,m} x_1^n x_2^m bp^n q^m d = \sum_{n,m} b(x_1 p)^n (q x_2)^m d \\ &= b(x_1 p)^*(q x_2)^* d \quad \in \mathcal{R}X^* \otimes_{\mathcal{R}} C_2. \end{aligned}$$

With $C_2' := C_2/\{db + qp \leq 1\}$ this can be improved to

$$\mathcal{C}X^* \simeq Z_{C_2'}(\mathcal{R}X^* \otimes_{\mathcal{R}} C_2') \quad \text{and} \quad \mathcal{C}M \simeq Z_{C_2'}(\mathcal{R}M \otimes_{\mathcal{R}} C_2').$$

To be extended to $Q_{\mathcal{R}}^{\mathcal{C}} : D\mathcal{R} \to D\mathcal{C}$.

Goal: regular expressions (over a non-free KA) for all CFLs.

# References


Hans Bekić.
Definable operations in general algebras, and the theory of automata and flowcharts.
In *Programming Languages and Their Definition, LNCS 177*, pages 30–55, Berlin, Heidelberg, 1984. Springer Verlag.

Niels Grathwohl, Fritz Henglein, and Dexter Kozen.
Infinitary axiomatization of the equational theory of context-free languages.
In *Fixed Points in Computer Science (FICS 2013)*, volume 126 of *EPTCS*, pages 44–55, 2013. doi:10.4204/EPTCS.126.4.

Mark Hopkins.
The Algebraic Approach I+II: The Algebraization of the Chomsky Hierarchy + Dioids, Quantales and Monads.
In *Relational Methods in Computer Science/Applications of Kleene Algebra*, Springer LNCS 4988, pages 155–190, 2008.

Dexter Kozen.
On induction vs. *-continuity.
In *Proc. Workshop on Logics of Programs 1981*, Springer LNCS 131, pages 167–176, 1981.

Dexter Kozen.
A completeness theorem for Kleene algebras and the algebra of regular events.
*Information and Computation*, 110(2):366–390, 1994.

Hans Leiß and Mark Hopkins.
Coequalizers and tensor products for continuous idempotent semirings.
In *Submitted to RAMiCS'17*, 2018.

Hans Leiß.
The matrix ring of a $\mu$-continuous Chomsky algebra is $\mu$-continuous.
In *25th EACSL Annual Conference on Computer Science Logic (CSL 2016)*, Leibniz International Proceedings in Informatics, pages 1–16. Dagstuhl Publishing, 2016.